98. I'm using a value for t and a point on the unit circle corresponding to t for which $\sin t = -\frac{\sqrt{10}}{2}$.

99. Because $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, I can conclude that $\cos\left(-\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$.

100. I can rewrite $\tan t$ as $\frac{1}{\cot t}$, as well as $\frac{\sin t}{\cos t}$.

101. If $\pi < t < \frac{3\pi}{2}$, which of the following is true?

- **a.** $\sin t > 0$ and $\tan t > 0$.
- **b.** $\sin t < 0$ and $\tan t < 0$.
- **c.** $\tan t > 0$ and $\cot t > 0$.
- **d.** $\tan t < 0$ and $\cot t < 0$.

102. If $f(x) = \sin x$ and $f(a) = \frac{1}{4}$, find the value of

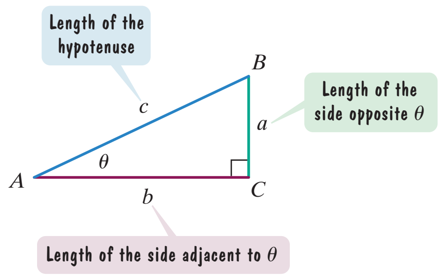
$$f(a) + f(a + 2\pi) + f(a + 4\pi) + f(a + 6\pi).$$

103. If $f(x) = \sin x$ and $f(a) = \frac{1}{4}$, find the value of $f(a) + 2f(-a)$.

104. The seats of a Ferris wheel are 40 feet from the wheel's center. When you get on the ride, your seat is 5 feet above the ground. How far above the ground are you after rotating through an angle of $\frac{17\pi}{4}$ radians? Round to the nearest foot.

Preview Exercises

Exercises 105–107 will help you prepare for the material covered in the next section. In each exercise, let θ be an acute angle in a right triangle, as shown in the figure. These exercises require the use of the Pythagorean Theorem.



105. If $a = 5$ and $b = 12$, find the ratio of the length of the side opposite θ to the length of the hypotenuse.

106. If $a = 1$ and $b = 1$, find the ratio of the length of the side opposite θ to the length of the hypotenuse. Simplify the ratio by rationalizing the denominator.

107. Simplify: $\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2$.

Section 4.3 Right Triangle Trigonometry

Objectives

1. Use right triangles to evaluate trigonometric functions.
2. Find function values for $30°\left(\frac{\pi}{6}\right)$, $45°\left(\frac{\pi}{4}\right)$, and $60°\left(\frac{\pi}{3}\right)$.
3. Use equal cofunctions of complements.
4. Use right triangle trigonometry to solve applied problems.

In the last century, Ang Rita Sherpa climbed Mount Everest ten times, all without the use of bottled oxygen.

Mountain climbers have forever been fascinated by reaching the top of Mount Everest, sometimes with tragic results. The mountain, on Asia's Tibet-Nepal border, is Earth's highest, peaking at an incredible 29,035 feet. The heights of mountains can be found using trigonometric functions. Remember that the word "*trigonometry*" means "*measurement of triangles*." Trigonometry is used in navigation, building, and engineering. For centuries, Muslims used trigonometry and the stars to navigate across the Arabian desert to Mecca, the birthplace of the prophet Muhammad, the founder of Islam. The ancient Greeks used trigonometry to record the locations of thousands of stars and worked out the motion of the Moon relative to Earth. Today, trigonometry is used to study the structure of DNA, the master molecule that determines how we grow from a single cell to a complex, fully developed adult.

1 Use right triangles to evaluate trigonometric functions.

Right Triangle Definitions of Trigonometric Functions

We have seen that in a unit circle, the radian measure of a central angle is equal to the measure of the intercepted arc. Thus, the value of a trigonometric function at the real number t is its value at an angle of t radians.

Figure 4.29(a) shows a central angle that measures $\frac{\pi}{3}$ radians and an intercepted arc of length $\frac{\pi}{3}$. Interpret $\frac{\pi}{3}$ as the measure of the central angle. In **Figure 4.29(b)**, we construct a right triangle by dropping a line segment from point P perpendicular to the x-axis.

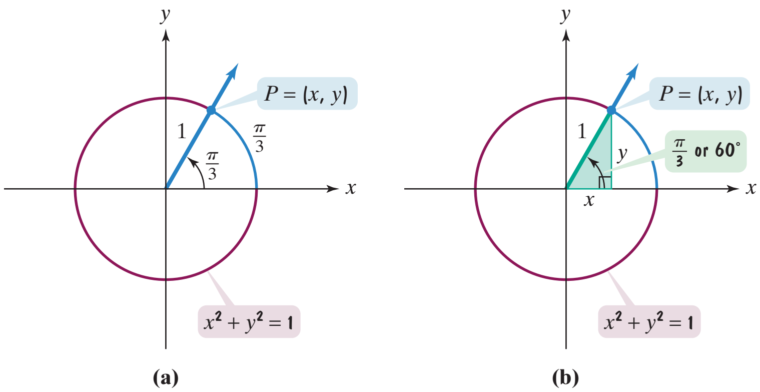


Figure 4.29
Interpreting trigonometric functions using a unit circle and a right triangle

Now we can think of $\frac{\pi}{3}$, or 60°, as the measure of an acute angle in the right triangle in **Figure 4.29(b)**. Because $\sin t$ is the second coordinate of point P and $\cos t$ is the first coordinate of point P, we see that

$$\sin\frac{\pi}{3} = \sin 60° = y = \frac{y}{1}$$

This is the length of the side opposite the 60° angle in the right triangle.

This is the length of the hypotenuse in the right triangle.

$$\cos\frac{\pi}{3} = \cos 60° = x = \frac{x}{1}.$$

This is the length of the side adjacent to the 60° angle in the right triangle.

This is the length of the hypotenuse in the right triangle.

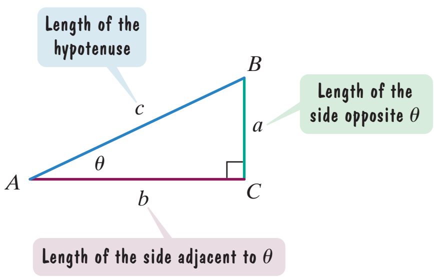


Figure 4.30

In solving certain kinds of problems, it is helpful to interpret trigonometric functions in right triangles, where angles are limited to acute angles. **Figure 4.30** shows a right triangle with one of its acute angles labeled θ. The side opposite the right angle, the hypotenuse, has length c. The other sides of the triangle are described by their position relative to the acute angle θ. One side is opposite θ. The length of this side is a. One side is adjacent to θ. The length of this side is b.

Right Triangle Definitions of Trigonometric Functions

See **Figure 4.30**. The six **trigonometric functions of the acute angle θ** are defined as follows:

$$\sin\theta = \frac{\text{length of side opposite angle }\theta}{\text{length of hypotenuse}} = \frac{a}{c} \qquad \csc\theta = \frac{\text{length of hypotenuse}}{\text{length of side opposite angle }\theta} = \frac{c}{a}$$

$$\cos\theta = \frac{\text{length of side adjacent to angle }\theta}{\text{length of hypotenuse}} = \frac{b}{c} \qquad \sec\theta = \frac{\text{length of hypotenuse}}{\text{length of side adjacent to angle }\theta} = \frac{c}{b}$$

$$\tan\theta = \frac{\text{length of side opposite angle }\theta}{\text{length of side adjacent to angle }\theta} = \frac{a}{b} \qquad \cot\theta = \frac{\text{length of side adjacent to angle }\theta}{\text{length of side opposite angle }\theta} = \frac{b}{a}$$

Each of the trigonometric functions of the acute angle θ is positive. Observe that the ratios in the second column in the box are the reciprocals of the corresponding ratios in the first column.

Study Tip

The word

SOHCAHTOA (pronounced: so-cah-tow-ah)

is a way to remember the right triangle definitions of the three basic trigonometric functions, sine, cosine, and tangent.

S	O H	C	A H	T	O A
↑	$\frac{\text{opp}}{\text{hyp}}$	↑	$\frac{\text{adj}}{\text{hyp}}$	↑	$\frac{\text{opp}}{\text{adj}}$
Sine		Cosine		Tangent	

"Some Old Hog Came Around Here and Took Our Apples."

Figure 4.31 shows four right triangles of varying sizes. In each of the triangles, θ is the same acute angle, measuring approximately 56.3°. All four of these similar triangles have the same shape and the lengths of corresponding sides are in the same ratio. In each triangle, the tangent function has the same value for the angle θ: $\tan\theta = \frac{3}{2}$.

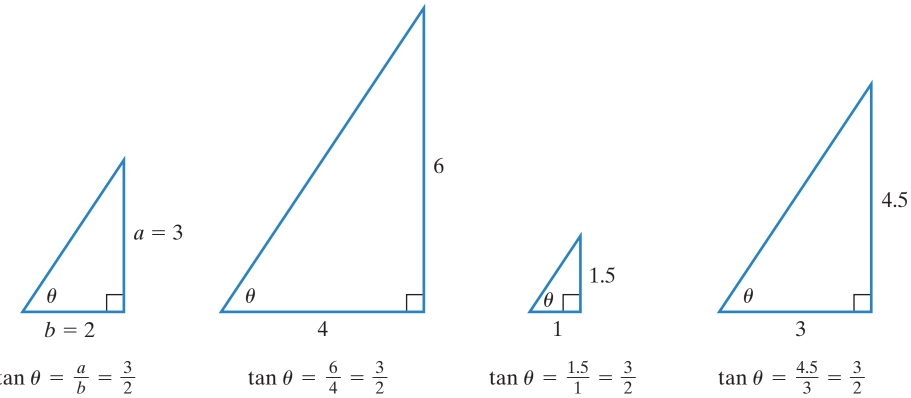


Figure 4.31 A particular acute angle always gives the same ratio of opposite to adjacent sides.

In general, **the trigonometric function values of θ depend only on the size of angle θ and not on the size of the triangle**.

EXAMPLE 1 Evaluating Trigonometric Functions

Find the value of each of the six trigonometric functions of θ in **Figure 4.32**.

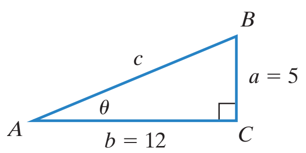


Figure 4.32

Solution We need to find the values of the six trigonometric functions of θ. However, we must know the lengths of all three sides of the triangle (a, b, and c) to evaluate all six functions. The values of a and b are given. We can use the Pythagorean Theorem, $c^2 = a^2 + b^2$, to find c.

$a = 5$ $\quad$ $b = 12$

$$c^2 = a^2 + b^2 = 5^2 + 12^2 = 25 + 144 = 169$$

$$c = \sqrt{169} = 13$$

Now that we know the lengths of the three sides of the triangle, we apply the definitions of the six trigonometric functions of θ. Referring to these lengths as opposite, adjacent, and hypotenuse, we have

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{5}{13} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{13}{5}$$

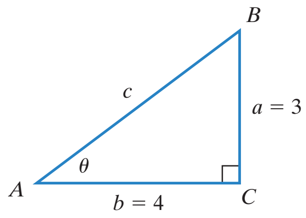
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{12}{13} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{13}{12}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{5}{12} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{12}{5}.$$

Study Tip

The function values in the second column are reciprocals of those in the first column. You can obtain these values by exchanging the numerator and denominator of the corresponding ratios in the first column.

Check Point 1 Find the value of each of the six trigonometric functions of θ in the figure.



EXAMPLE 2 Evaluating Trigonometric Functions

Find the value of each of the six trigonometric functions of θ in **Figure 4.33**.

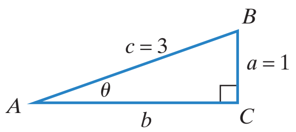


Figure 4.33

Solution We begin by finding b.

$$a^2 + b^2 = c^2 \quad \text{Use the Pythagorean Theorem.}$$
$$1^2 + b^2 = 3^2 \quad \text{Figure 4.33 shows that } a = 1 \text{ and } c = 3.$$
$$1 + b^2 = 9 \quad 1^2 = 1 \text{ and } 3^2 = 9.$$
$$b^2 = 8 \quad \text{Subtract 1 from both sides.}$$
$$b = \sqrt{8} = 2\sqrt{2} \quad \text{Take the principal square root and simplify: } \sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4}\sqrt{2} = 2\sqrt{2}.$$

Now that we know the lengths of the three sides of the triangle, we apply the definitions of the six trigonometric functions of θ.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{3} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{3}{1} = 3$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{2\sqrt{2}}{3} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{3}{2\sqrt{2}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{2\sqrt{2}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{2\sqrt{2}}{1} = 2\sqrt{2}$$

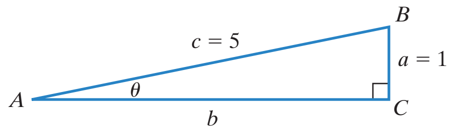
We can simplify the values of $\tan\theta$ and $\sec\theta$ by rationalizing the denominators:

$$\tan\theta = \frac{1}{2\sqrt{2}} = \frac{1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2 \cdot 2} = \frac{\sqrt{2}}{4} \qquad \sec\theta = \frac{3}{2\sqrt{2}} = \frac{3}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{3\sqrt{2}}{2 \cdot 2} = \frac{3\sqrt{2}}{4}.$$

We are multiplying by 1 and not changing the value of $\frac{1}{2\sqrt{2}}$.

We are multiplying by 1 and not changing the value of $\frac{3}{2\sqrt{2}}$.

Check Point 2 Find the value of each of the six trigonometric functions of θ in the figure. Express each value in simplified form.



2 Find function values for $30°\left(\frac{\pi}{6}\right)$, $45°\left(\frac{\pi}{4}\right)$, and $60°\left(\frac{\pi}{3}\right)$.

Function Values for Some Special Angles

In Section 4.2, we used the unit circle to find values of the trigonometric functions at $\frac{\pi}{4}$. How can we find the values of the trigonometric functions at $\frac{\pi}{4}$, or 45°, using a right triangle? We construct a right triangle with a 45° angle, as shown in **Figure 4.34** at the top of the next page. The triangle actually has two 45° angles. Thus, the triangle is isosceles—that is, it has two sides of the same length. Assume that each leg of the triangle has a length equal to 1. We can find the length of the hypotenuse using the Pythagorean Theorem.

$$(\text{length of hypotenuse})^2 = 1^2 + 1^2 = 2$$
$$\text{length of hypotenuse} = \sqrt{2}$$

With **Figure 4.34**, we can determine the trigonometric function values for 45°.

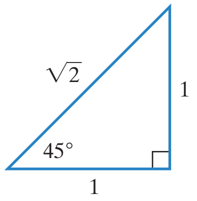


Figure 4.34 An isosceles right triangle

EXAMPLE 3 Evaluating Trigonometric Functions of 45°

Use **Figure 4.34** to find sin 45°, cos 45°, and tan 45°.

Solution We apply the definitions of these three trigonometric functions. Where appropriate, we simplify by rationalizing denominators.

$$\sin 45° = \frac{\text{length of side opposite } 45°}{\text{length of hypotenuse}} = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Rationalize denominators.

$$\cos 45° = \frac{\text{length of side adjacent to } 45°}{\text{length of hypotenuse}} = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45° = \frac{\text{length of side opposite } 45°}{\text{length of side adjacent to } 45°} = \frac{1}{1} = 1$$

Check Point 3 Use **Figure 4.34** to find csc 45°, sec 45°, and cot 45°.

When you worked Check Point 3, did you actually use **Figure 4.34** or did you use reciprocals to find the values?

$$\csc 45° = \sqrt{2} \qquad \sec 45° = \sqrt{2} \qquad \cot 45° = 1$$

Take the reciprocal of $\sin 45° = \frac{1}{\sqrt{2}}$.

Take the reciprocal of $\cos 45° = \frac{1}{\sqrt{2}}$.

Take the reciprocal of $\tan 45° = \frac{1}{1}$.

Notice that if you use reciprocals, you should take the reciprocal of a function value before the denominator is rationalized. In this way, the reciprocal value will not contain a radical in the denominator.

Two other angles that occur frequently in trigonometry are 30°, or $\frac{\pi}{6}$ radian, and 60°, or $\frac{\pi}{3}$ radian, angles. We can find the values of the trigonometric functions of 30° and 60° by using a right triangle. To form this right triangle, draw an equilateral triangle—that is a triangle with all sides the same length. Assume that each side has a length equal to 2. Now take half of the equilateral triangle. We obtain the right triangle in **Figure 4.35**. This right triangle has a hypotenuse of length 2 and a leg of length 1. The other leg has length a, which can be found using the Pythagorean Theorem.

$$a^2 + 1^2 = 2^2$$
$$a^2 + 1 = 4$$
$$a^2 = 3$$
$$a = \sqrt{3}$$

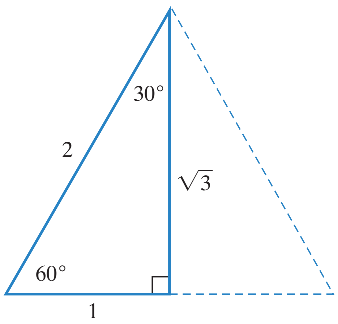


Figure 4.35 30°–60°–90° triangle

With the right triangle in **Figure 4.35**, we can determine the trigonometric functions for 30° and 60°.

EXAMPLE 4 Evaluating Trigonometric Functions of 30° and 60°

Use **Figure 4.35** to find sin 60°, cos 60°, sin 30°, and cos 30°.

Solution We begin with 60°. Use the angle on the lower left in **Figure 4.35**.

$$\sin 60° = \frac{\text{length of side opposite } 60°}{\text{length of hypotenuse}} = \frac{\sqrt{3}}{2}$$

$$\cos 60° = \frac{\text{length of side adjacent to } 60°}{\text{length of hypotenuse}} = \frac{1}{2}$$

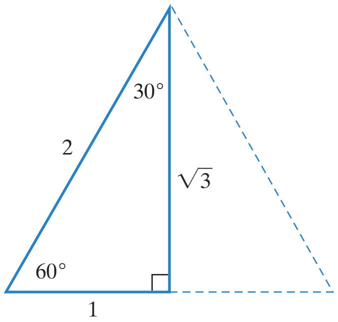


Figure 4.35 (repeated)

To find sin 30° and cos 30°, use the angle on the upper right in **Figure 4.35**.

$$\sin 30^\circ = \frac{\text{length of side opposite } 30^\circ}{\text{length of hypotenuse}} = \frac{1}{2}$$

$$\cos 30^\circ = \frac{\text{length of side adjacent to } 30^\circ}{\text{length of hypotenuse}} = \frac{\sqrt{3}}{2}$$

Check Point 4 Use **Figure 4.35** to find tan 60° and tan 30°. If a radical appears in a denominator, rationalize the denominator.

Because we will often use the function values of 30°, 45°, and 60°, you should learn to construct the right triangles shown in **Figures 4.34** and **4.35**. With sufficient practice, you will memorize the values in **Table 4.2**.

Table 4.2 Trigonometric Functions of Special Angles

θ	$30^\circ = \frac{\pi}{6}$	$45^\circ = \frac{\pi}{4}$	$60^\circ = \frac{\pi}{3}$
$\sin\theta$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan\theta$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

3 Use equal cofunctions of complements.

Trigonometric Functions and Complements

Two positive angles are **complements** if their sum is 90° or $\frac{\pi}{2}$. For example, angles of 70° and 20° are complements because $70^\circ + 20^\circ = 90^\circ$.

In Section 4.2, we used the unit circle to establish fundamental trigonometric identities. Another relationship among trigonometric functions is based on angles that are complements. Refer to **Figure 4.36**. Because the sum of the angles of any triangle is 180°, in a right triangle the sum of the acute angles is 90°. Thus, the acute angles are complements. If the degree measure of one acute angle is θ, then the degree measure of the other acute angle is $(90^\circ - \theta)$. This angle is shown on the upper right in **Figure 4.36**.

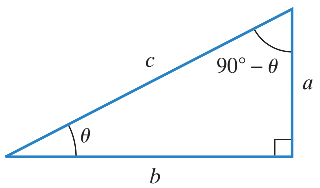


Figure 4.36

Let's use **Figure 4.36** to compare $\sin\theta$ and $\cos(90^\circ - \theta)$.

$$\sin\theta = \frac{\text{length of side opposite } \theta}{\text{length of hypotenuse}} = \frac{a}{c}$$

$$\cos(90^\circ - \theta) = \frac{\text{length of side adjacent to } (90^\circ - \theta)}{\text{length of hypotenuse}} = \frac{a}{c}$$

Thus, $\sin\theta = \cos(90^\circ - \theta)$. If two angles are complements, the sine of one equals the cosine of the other. Because of this relationship, the sine and cosine are called *cofunctions* of each other. The name *cosine* is a shortened form of the phrase *complement's sine*.

Any pair of trigonometric functions f and g for which

$$f(\theta) = g(90^\circ - \theta) \quad \text{and} \quad g(\theta) = f(90^\circ - \theta)$$

are called **cofunctions**. Using **Figure 4.36**, we can show that the tangent and cotangent are also cofunctions of each other. So are the secant and cosecant.

Cofunction Identities

The value of a trigonometric function of θ is equal to the cofunction of the complement of θ. Cofunctions of complementary angles are equal.

$$\sin\theta = \cos(90° - \theta) \qquad \cos\theta = \sin(90° - \theta)$$
$$\tan\theta = \cot(90° - \theta) \qquad \cot\theta = \tan(90° - \theta)$$
$$\sec\theta = \csc(90° - \theta) \qquad \csc\theta = \sec(90° - \theta)$$

If θ is in radians, replace 90° with $\frac{\pi}{2}$.

EXAMPLE 5 Using Cofunction Identities

Find a cofunction with the same value as the given expression:

a. $\sin 72°$ **b.** $\csc\frac{\pi}{3}$.

Solution Because the value of a trigonometric function of θ is equal to the cofunction of the complement of θ, we need to find the complement of each angle. We do this by subtracting the angle's measure from 90° or its radian equivalent, $\frac{\pi}{2}$.

a. $\sin 72° = \cos(90° - 72°) = \cos 18°$

We have a function and its cofunction.

b. $\csc\frac{\pi}{3} = \sec\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \sec\left(\frac{3\pi}{6} - \frac{2\pi}{6}\right) = \sec\frac{\pi}{6}$

We have a cofunction and its function.

Perform the subtraction using the least common denominator, 6.

Check Point 5 Find a cofunction with the same value as the given expression:

a. $\sin 46°$ **b.** $\cot\frac{\pi}{12}$.

4 Use right triangle trigonometry to solve applied problems.

Applications

Many applications of right triangle trigonometry involve the angle made with an imaginary horizontal line. As shown in **Figure 4.37**, an angle formed by a horizontal line and the line of sight to an object that is above the horizontal line is called the **angle of elevation**. The angle formed by a horizontal line and the line of sight to an object that is below the horizontal line is called the **angle of depression**. Transits and sextants are instruments used to measure such angles.

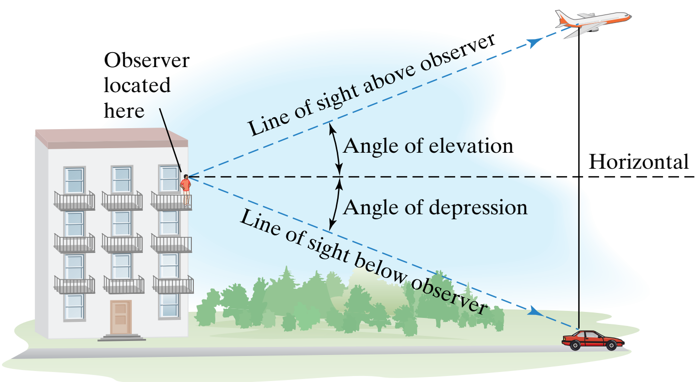


Figure 4.37

EXAMPLE 6 Problem Solving Using an Angle of Elevation

Sighting the top of a building, a surveyor measured the angle of elevation to be 22°. The transit is 5 feet above the ground and 300 feet from the building. Find the building's height.

Solution The situation is illustrated in **Figure 4.38**. Let a be the height of the portion of the building that lies above the transit. The height of the building is the transit's height, 5 feet, plus a. Thus, we need to identify a trigonometric function that will make it possible to find a. In terms of the 22° angle, we are looking for the side

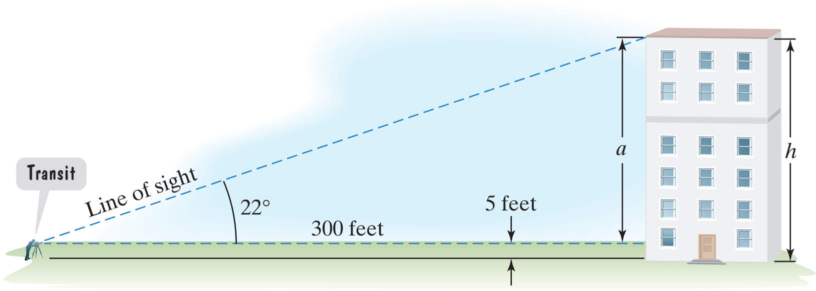


Figure 4.38

opposite the angle. The transit is 300 feet from the building, so the side adjacent to the 22° angle is 300 feet. Because we have a known angle, an unknown opposite side, and a known adjacent side, we select the tangent function.

$$\tan 22^\circ = \frac{a}{300}$$

Length of side opposite the 22° angle

Length of side adjacent to the 22° angle

$a = 300 \tan 22^\circ$ Multiply both sides of the equation by 300.

$a \approx 121$ Use a calculator in the degree mode.

The height of the part of the building above the transit is approximately 121 feet. Thus, the height of the building is determined by adding the transit's height, 5 feet, to 121 feet.

$$h \approx 5 + 121 = 126$$

The building's height is approximately 126 feet.

Check Point 6 The irregular blue shape in **Figure 4.39** represents a lake. The distance across the lake, a, is unknown. To find this distance, a surveyor took the measurements shown in the figure. What is the distance across the lake?

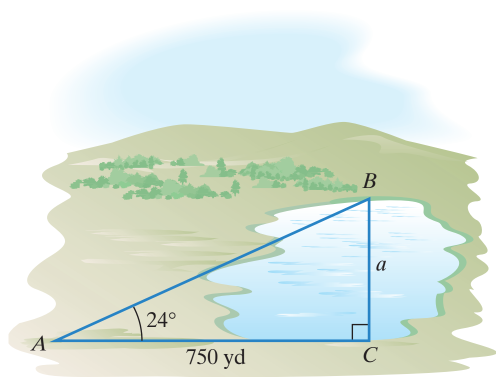


Figure 4.39

If two sides of a right triangle are known, an appropriate trigonometric function can be used to find an acute angle θ in the triangle. You will also need to use an inverse trigonometric key on a calculator. These keys use a function value to display the acute angle θ. For example, suppose that $\sin \theta = 0.866$. We can find θ in the degree mode by using the secondary *inverse sine* key, usually labeled $\boxed{\text{SIN}^{-1}}$. The key $\boxed{\text{SIN}^{-1}}$ is not a button you will actually press. It is the secondary function for the button labeled $\boxed{\text{SIN}}$.

Pressing $\boxed{\text{2nd}}\boxed{\text{SIN}}$ accesses the inverse sine key, $\boxed{\text{SIN}^{-1}}$.

Many Scientific Calculators:

.866 $\boxed{\text{2nd}}\boxed{\text{SIN}}$

Many Graphing Calculators:

$\boxed{\text{2nd}}\boxed{\text{SIN}}$ .866 $\boxed{\text{ENTER}}$

The display should show approximately 59.99, which can be rounded to 60. Thus, if $\sin\theta = 0.866$ and θ is acute, then $\theta \approx 60°$.

EXAMPLE 7 Determining the Angle of Elevation

A building that is 21 meters tall casts a shadow 25 meters long. Find the angle of elevation of the sun to the nearest degree.

Solution The situation is illustrated in **Figure 4.40**. We are asked to find θ. We begin with the tangent function.

$$\tan\theta = \frac{\text{side opposite } \theta}{\text{side adjacent to } \theta} = \frac{21}{25}$$

We use a calculator in the degree mode to find θ.

Pressing $\boxed{\text{2nd}}\boxed{\text{TAN}}$ accesses the inverse tangent key, $\boxed{\text{TAN}^{-1}}$.

Many Scientific Calculators:

$\boxed{(}\,21\boxed{\div}25\boxed{)}\boxed{\text{2nd}}\boxed{\text{TAN}}$

Many Graphing Calculators:

$\boxed{\text{2nd}}\boxed{\text{TAN}}\boxed{(}\,21\boxed{\div}25\boxed{)}\boxed{\text{ENTER}}$

The display should show approximately 40. Thus, the angle of elevation of the sun is approximately 40°.

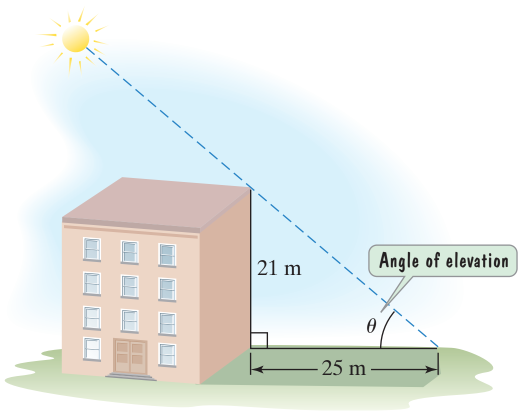


Figure 4.40

Check Point 7 A flagpole that is 14 meters tall casts a shadow 10 meters long. Find the angle of elevation of the sun to the nearest degree.

The Mountain Man

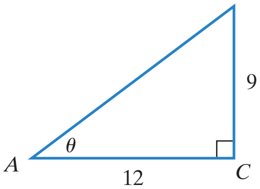
In the 1930s, a *National Geographic* team headed by Brad Washburn used trigonometry to create a map of the 5000-square-mile region of the Yukon, near the Canadian border. The team started with aerial photography. By drawing a network of angles on the photographs, the approximate locations of the major mountains and their rough heights were determined. The expedition then spent three months on foot to find the exact heights. Team members established two base points a known distance apart, one directly under the mountain's peak. By measuring the angle of elevation from one of the base points to the peak, the tangent function was used to determine the peak's height. The Yukon expedition was a major advance in the way maps are made.

Exercise Set 4.3

Practice Exercises

In Exercises 1–8, use the Pythagorean Theorem to find the length of the missing side of each right triangle. Then find the value of each of the six trigonometric functions of θ.

1.



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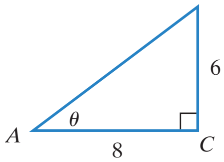

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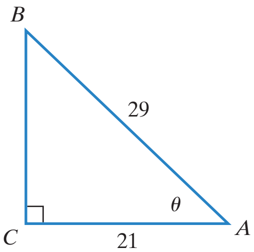

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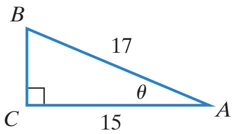

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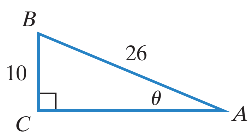

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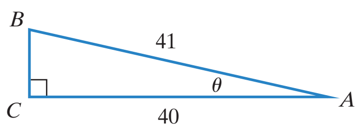

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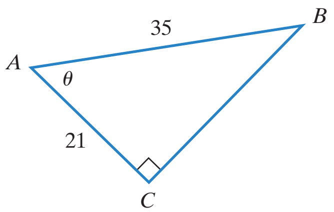

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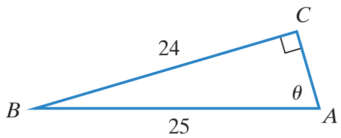
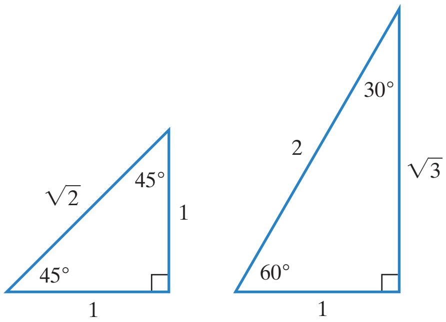


In Exercises 9–20, use the given triangles to evaluate each expression. If necessary, express the value without a square root in the denominator by rationalizing the denominator.



9. $\cos 30°$

10. $\tan 30°$

11. $\sec 45°$

12. $\csc 45°$

13. $\tan \frac{\pi}{3}$

14. $\cot \frac{\pi}{3}$

15. $\sin \frac{\pi}{4} - \cos \frac{\pi}{4}$

16. $\tan \frac{\pi}{4} + \csc \frac{\pi}{6}$

17. $\sin \frac{\pi}{3} \cos \frac{\pi}{4} - \tan \frac{\pi}{4}$

18. $\cos \frac{\pi}{3} \sec \frac{\pi}{3} - \cot \frac{\pi}{3}$

19. $2 \tan \frac{\pi}{3} + \cos \frac{\pi}{4} \tan \frac{\pi}{6}$

20. $6 \tan \frac{\pi}{4} + \sin \frac{\pi}{3} \sec \frac{\pi}{6}$

In Exercises 21–28, find a cofunction with the same value as the given expression.

21. $\sin 7°$

22. $\sin 19°$

23. $\csc 25°$

24. $\csc 35°$

25. $\tan \frac{\pi}{9}$

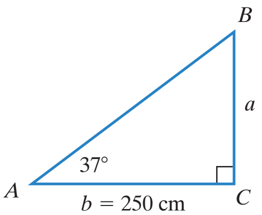
26. $\tan \frac{\pi}{7}$

27. $\cos \frac{2\pi}{5}$

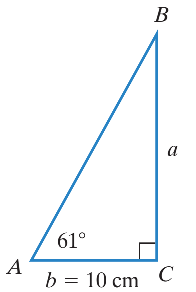
28. $\cos \frac{3\pi}{8}$

In Exercises 29–34, find the measure of the side of the right triangle whose length is designated by a lowercase letter. Round answers to the nearest whole number.

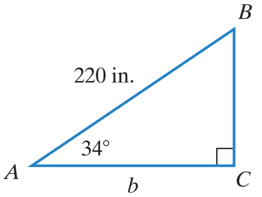
29.



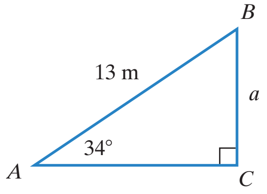
30.



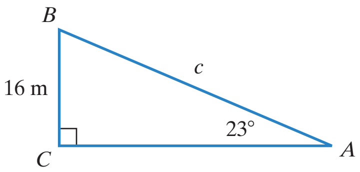
31.



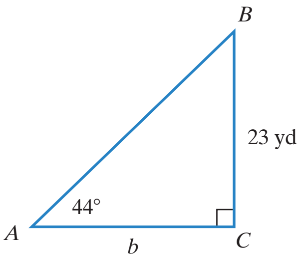
32.



33.



34.



In Exercises 35–38, use a calculator to find the value of the acute angle θ to the nearest degree.

35. $\sin\theta = 0.2974$

36. $\cos\theta = 0.8771$

37. $\tan\theta = 4.6252$

38. $\tan\theta = 26.0307$

In Exercises 39–42, use a calculator to find the value of the acute angle θ in radians, rounded to three decimal places.

39. $\cos\theta = 0.4112$

40. $\sin\theta = 0.9499$

41. $\tan\theta = 0.4169$

42. $\tan\theta = 0.5117$

Practice Plus

In Exercises 43–48, find the exact value of each expression. Do not use a calculator.

43. $\dfrac{\tan\frac{\pi}{3}}{2} - \dfrac{1}{\sec\frac{\pi}{6}}$

44. $\dfrac{1}{\cot\frac{\pi}{4}} - \dfrac{2}{\csc\frac{\pi}{6}}$

45. $1 + \sin^2 40° + \sin^2 50°$

46. $1 - \tan^2 10° + \csc^2 80°$

47. $\csc 37° \sec 53° - \tan 53° \cot 37°$

48. $\cos 12° \sin 78° + \cos 78° \sin 12°$

In Exercises 49–50, express each exact value as a single fraction. Do not use a calculator.

49. If $f(\theta) = 2\cos\theta - \cos 2\theta$, find $f\left(\frac{\pi}{6}\right)$.

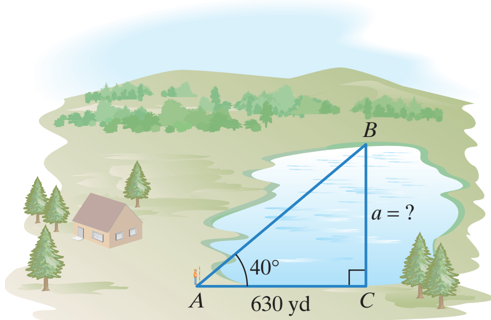
50. If $f(\theta) = 2\sin\theta - \sin\frac{\theta}{2}$, find $f\left(\frac{\pi}{3}\right)$.

51. If θ is an acute angle and $\cot\theta = \frac{1}{4}$, find $\tan\left(\frac{\pi}{2} - \theta\right)$.

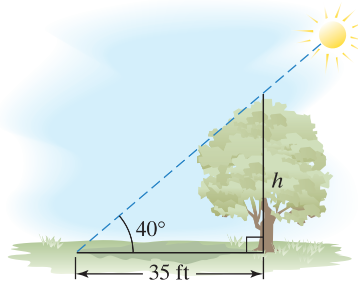
52. If θ is an acute angle and $\cos\theta = \frac{1}{3}$, find $\csc\left(\frac{\pi}{2} - \theta\right)$.

Application Exercises

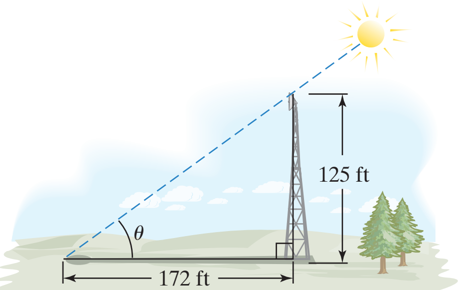
53. To find the distance across a lake, a surveyor took the measurements shown in the figure. Use these measurements to determine how far it is across the lake. Round to the nearest yard.



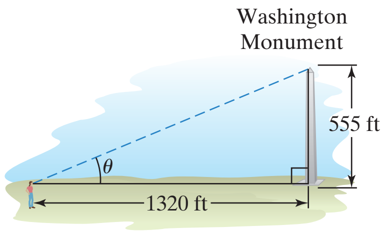
54. At a certain time of day, the angle of elevation of the sun is 40°. To the nearest foot, find the height of a tree whose shadow is 35 feet long.



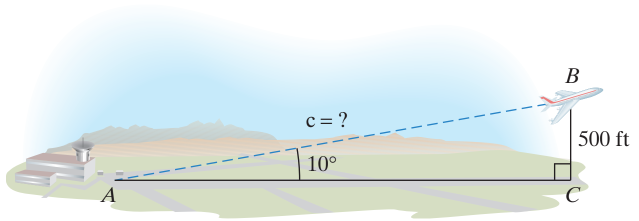
55. A tower that is 125 feet tall casts a shadow 172 feet long. Find the angle of elevation of the sun to the nearest degree.



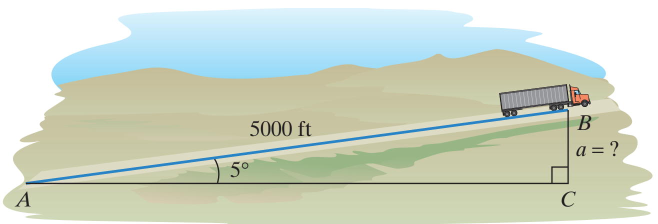
56. The Washington Monument is 555 feet high. If you stand one quarter of a mile, or 1320 feet, from the base of the monument and look to the top, find the angle of elevation to the nearest degree.



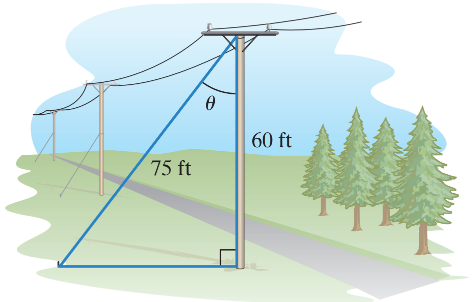
57. A plane rises from take-off and flies at an angle of 10° with the horizontal runway. When it has gained 500 feet, find the distance, to the nearest foot, the plane has flown.



58. A road is inclined at an angle of 5°. After driving 5000 feet along this road, find the driver's increase in altitude. Round to the nearest foot.



59. A telephone pole is 60 feet tall. A guy wire 75 feet long is attached from the ground to the top of the pole. Find the angle between the wire and the pole to the nearest degree.



60. A telephone pole is 55 feet tall. A guy wire 80 feet long is attached from the ground to the top of the pole. Find the angle between the wire and the pole to the nearest degree.

Writing in Mathematics

61. If you are given the lengths of the sides of a right triangle, describe how to find the sine of either acute angle.

62. Describe one similarity and one difference between the definitions of $\sin\theta$ and $\cos\theta$, where θ is an acute angle of a right triangle.

63. Describe the triangle used to find the trigonometric functions of 45°.

64. Describe the triangle used to find the trigonometric functions of 30° and 60°.

65. Describe a relationship among trigonometric functions that is based on angles that are complements.

66. Describe what is meant by an angle of elevation and an angle of depression.

67. Stonehenge, the famous "stone circle" in England, was built between 2750 B.C. and 1300 B.C. using solid stone blocks weighing over 99,000 pounds each. It required 550 people to pull a single stone up a ramp inclined at a 9° angle. Describe how right triangle trigonometry can be used to determine the distance the 550 workers had to drag a stone in order to raise it to a height of 30 feet.

Technology Exercises

68. Use a calculator in the radian mode to fill in the values in the following table. Then draw a conclusion about $\frac{\sin\theta}{\theta}$ as θ approaches 0.

θ	0.4	0.3	0.2	0.1	0.01	0.001	0.0001	0.00001
$\sin\theta$								
$\frac{\sin\theta}{\theta}$								

69. Use a calculator in the radian mode to fill in the values in the following table. Then draw a conclusion about $\frac{\cos\theta - 1}{\theta}$ as θ approaches 0.

θ	0.4	0.3	0.2	0.1	0.01	0.001	0.0001	0.00001
$\cos\theta$								
$\frac{\cos\theta - 1}{\theta}$								

Critical Thinking Exercises

Make Sense? *In Exercises 70–73, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

70. For a given angle θ, I found a slight increase in $\sin\theta$ as the size of the triangle increased.

71. Although I can use an isosceles right triangle to determine the exact value of $\sin\frac{\pi}{4}$, I can also use my calculator to obtain this value.

72. The sine and cosine are cofunctions and reciprocals of each other.

73. Standing under this arch, I can determine its height by measuring the angle of elevation to the top of the arch and my distance to a point directly under the arch.

Delicate Arch in Arches National Park, Utah

In Exercises 74–77, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

74. $\frac{\tan 45°}{\tan 15°} = \tan 3°$

75. $\tan^2 15° - \sec^2 15° = -1$

76. $\sin 45° + \cos 45° = 1$

77. $\tan^2 5° = \tan 25°$

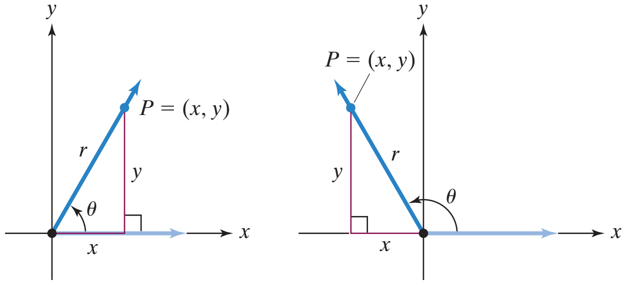
78. Explain why the sine or cosine of an acute angle cannot be greater than or equal to 1.

79. Describe what happens to the tangent of an acute angle as the angle gets close to 90°. What happens at 90°?

80. From the top of a 250-foot lighthouse, a plane is sighted overhead and a ship is observed directly below the plane. The angle of elevation of the plane is 22° and the angle of depression of the ship is 35°. Find **a.** the distance of the ship from the lighthouse; **b.** the plane's height above the water. Round to the nearest foot.

Preview Exercises

Exercises 81–83 will help you prepare for the material covered in the next section. Use these figures to solve Exercises 81–82.



(a) θ lies in quadrant I.

(b) θ lies in quadrant II.

81. a. Write a ratio that expresses $\sin\theta$ for the right triangle in **Figure (a)**.

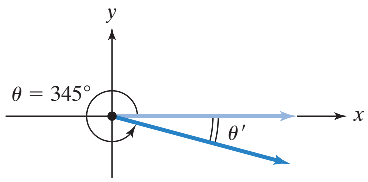
b. Determine the ratio that you wrote in part (a) for **Figure (b)** with $x = -3$ and $y = 4$. Is this ratio positive or negative?

82. a. Write a ratio that expresses $\cos\theta$ for the right triangle in **Figure (a)**.

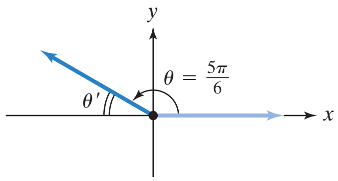
b. Determine the ratio that you wrote in part (a) for **Figure (b)** with $x = -3$ and $y = 5$. Is this ratio positive or negative?

83. Find the positive angle θ' formed by the terminal side of θ and the x-axis.

a.



b.



Section 4.4 Trigonometric Functions of Any Angle

Objectives

1. Use the definitions of trigonometric functions of any angle.
2. Use the signs of the trigonometric functions.
3. Find reference angles.
4. Use reference angles to evaluate trigonometric functions.

Cycles govern many aspects of life—heartbeats, sleep patterns, seasons, and tides all follow regular, predictable cycles. Because of their periodic nature, trigonometric functions are used to model phenomena that occur in cycles. It is helpful to apply these models regardless of whether we think of the domains of trigonomentric functions as sets of real numbers or sets of angles. In order to understand and use models for cyclic phenomena from an angle perspective, we need to move beyond right triangles.