Trigonometric Functions

4

Have you had days where your physical, intellectual, and emotional potentials were all at their peak? Then there are those other days when we feel we should not even bother getting out of bed. Do our potentials run in oscillating cycles like the tides? Can they be described mathematically? In this chapter, you will encounter functions that enable us to model phenomena that occur in cycles.

Graphs of functions showing a person's biorhythms, the physical, intellectual, and emotional cycles we experience in life, are presented in Exercises 75–82 of Exercise Set 4.5.

Section 4.1

Angles and Radian Measure

Objectives

1. Recognize and use the vocabulary of angles.
2. Use degree measure.
3. Use radian measure.
4. Convert between degrees and radians.
5. Draw angles in standard position.
6. Find coterminal angles.
7. Find the length of a circular arc.
8. Use linear and angular speed to describe motion on a circular path.

The San Francisco Museum of Modern Art was constructed in 1995 to illustrate how art and architecture can enrich one another. The exterior involves geometric shapes, symmetry, and unusual facades. Although there are no windows, natural light streams in through a truncated cylindrical skylight that crowns the building. The architect worked with a scale model of the museum at the site and observed how light hit it during different times of the day. These observations were used to cut the cylindrical skylight at an angle that maximizes sunlight entering the interior.

Angles play a critical role in creating modern architecture. They are also fundamental in trigonometry. In this section, we begin our study of trigonometry by looking at angles and methods for measuring them.

Angles

1 Recognize and use the vocabulary of angles.

The hour hand of a clock suggests a **ray**, a part of a line that has only one endpoint and extends forever in the opposite direction. An **angle** is formed by two rays that have a common endpoint. One ray is called the **initial side** and the other the **terminal side**.

A rotating ray is often a useful way to think about angles. The ray in **Figure 4.1** rotates from 12 to 2. The ray pointing to 12 is the **initial side** and the ray pointing to 2 is the **terminal side**. The common endpoint of an angle's initial side and terminal side is the **vertex** of the angle.

Figure 4.2 shows an angle. The arrow near the vertex shows the direction and the amount of rotation from the initial side to the terminal side. Several methods can be used to name an angle. Lowercase Greek letters, such as α (alpha), β (beta), γ (gamma), and θ (theta), are often used.

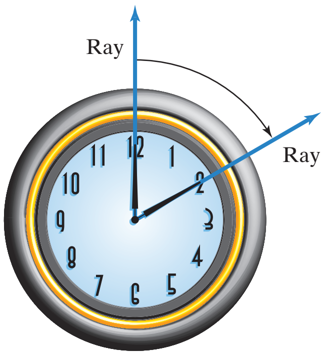


Figure 4.1 Clock with hands forming an angle

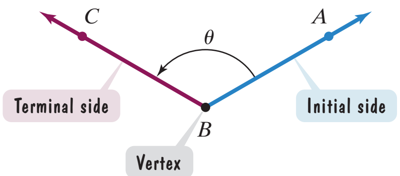


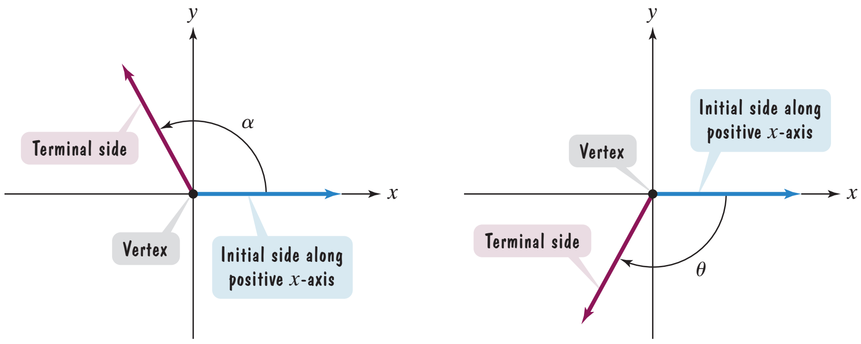
Figure 4.2 An angle; two rays with a common endpoint

An angle is in **standard position** if

- its vertex is at the origin of a rectangular coordinate system and
- its initial side lies along the positive x-axis.

The angles in **Figure 4.3** at the top of the next page are both in standard position.

When we see an initial side and a terminal side in place, there are two kinds of rotations that could have generated the angle. The arrow in **Figure 4.3(a)** indicates that the rotation from the initial side to the terminal side is in the counterclockwise direction. **Positive angles** are generated by counterclockwise rotation. Thus, angle α is positive. By contrast, the arrow in **Figure 4.3(b)** shows that the rotation from the



(a) α in standard position; α positive

(b) θ in standard position; θ negative

Figure 4.3 Two angles in standard position

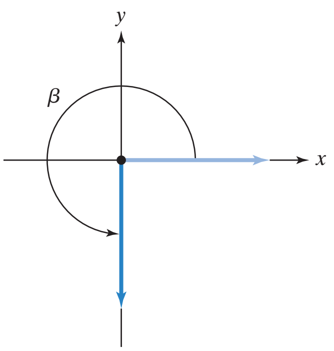


Figure 4.4 β is a quadrantal angle.

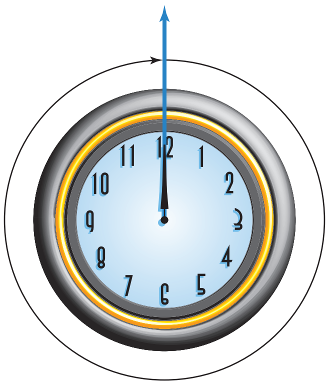
initial side to the terminal side is in the clockwise direction. **Negative angles** are generated by clockwise rotation. Thus, angle θ is negative.

When an angle is in standard position, its terminal side can lie in a quadrant. We say that the angle **lies in that quadrant**. For example, in **Figure 4.3(a)**, the terminal side of angle α lies in quadrant II. Thus, angle α lies in quadrant II. By contrast, in **Figure 4.3(b)**, the terminal side of angle θ lies in quadrant III. Thus, angle θ lies in quadrant III.

Must all angles in standard position lie in a quadrant? The answer is no. The terminal side can lie on the x-axis or the y-axis. For example, angle β in **Figure 4.4** has a terminal side that lies on the negative y-axis. An angle is called a **quadrantal angle** if its terminal side lies on the x-axis or on the y-axis. Angle β in **Figure 4.4** is an example of a quadrantal angle.

2 Use degree measure.

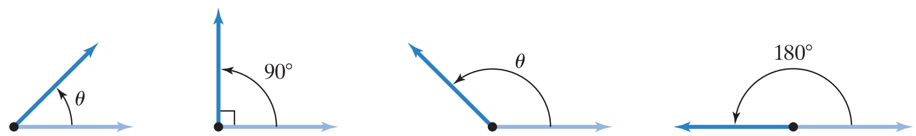
Measuring Angles Using Degrees



A complete 360° rotation

Angles are measured by determining the amount of rotation from the initial side to the terminal side. One way to measure angles is in **degrees**, symbolized by a small, raised circle °. Think of the hour hand of a clock. From 12 noon to 12 midnight, the hour hand moves around in a complete circle. By definition, the ray has rotated through 360 degrees, or 360°. Using 360° as the amount of rotation of a ray back onto itself, a degree, 1°, is $\frac{1}{360}$ of a complete rotation.

Figure 4.5 shows that certain angles have special names. An **acute angle** measures less than 90° [see **Figure 4.5(a)**]. A **right angle**, one quarter of a complete rotation, measures 90° [**Figure 4.5(b)**]. Examine the right angle—do you see a small square at the vertex? This symbol is used to indicate a right angle. An **obtuse angle** measures more than 90°, but less than 180° [**Figure 4.5(c)**]. Finally, a **straight angle**, one-half a complete rotation, measures 180° [**Figure 4.5(d)**].



(a) Acute angle $(0° < \theta < 90°)$

(b) Right angle $(\frac{1}{4}$ rotation)

(c) Obtuse angle $(90° < \theta < 180°)$

(d) Straight angle $(\frac{1}{2}$ rotation)

Figure 4.5 Classifying angles by their degree measurement

We will be using notation such as $\theta = 60°$ to refer to an angle θ whose measure is 60°. We also refer to *an angle of* 60° or a 60° *angle*, rather than using the more precise (but cumbersome) phrase *an angle whose measure is* 60°.

Technology

Fractional parts of degrees are measured in minutes and seconds.

One minute, written $1'$, is $\frac{1}{60}$ degree: $1' = \frac{1}{60}°$.

One second, written $1''$, is $\frac{1}{3600}$ degree: $1'' = \frac{1}{3600}°$.

For example,

$$31°47'12'' = \left(31 + \frac{47}{60} + \frac{12}{3600}\right)^{\circ} \approx 31.787°.$$

Many calculators have keys for changing an angle from degree-minute-second notation (D°M′S″) to a decimal form and vice versa.

3 Use radian measure.

Measuring Angles Using Radians

Another way to measure angles is in *radians*. Let's first define an angle measuring **1 radian**. We use a circle of radius r. In **Figure 4.6**, we've constructed an angle whose vertex is at the center of the circle. Such an angle is called a **central angle**. Notice that this central angle intercepts an arc along the circle measuring r units. The radius of the circle is also r units. The measure of such an angle is 1 radian.

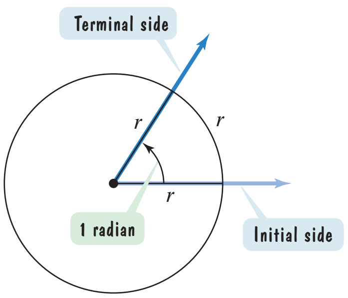


Figure 4.6 For a 1-radian angle, the intercepted arc and the radius are equal.

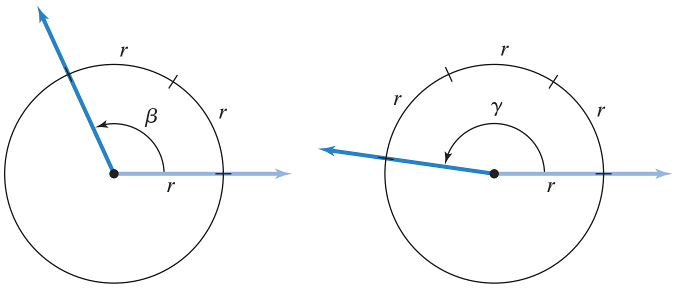
Definition of a Radian

One radian is the measure of the central angle of a circle that intercepts an arc equal in length to the radius of the circle.

The **radian measure** of any central angle is the length of the intercepted arc divided by the circle's radius. In **Figure 4.7(a)**, the length of the arc intercepted by angle β is double the radius, r. We find the measure of angle β in radians by dividing the length of the intercepted arc by the radius.

$$\beta = \frac{\text{length of the intercepted arc}}{\text{radius}} = \frac{2r}{r} = 2$$

Thus, angle β measures 2 radians.



(a) $\beta = 2$ radians **(b)** $\gamma = 3$ radians

Figure 4.7 Two central angles measured in radians

In **Figure 4.7(b)**, the length of the intercepted arc is triple the radius, r. Let us find the measure of angle γ:

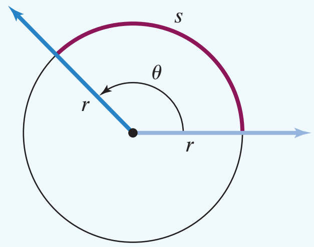
$$\gamma = \frac{\text{length of the intercepted arc}}{\text{radius}} = \frac{3r}{r} = 3.$$

Thus, angle γ measures 3 radians.

Radian Measure

Consider an arc of length s on a circle of radius r. The measure of the central angle, θ, that intercepts the arc is

$$\theta = \frac{s}{r} \text{ radians.}$$



EXAMPLE 1 Computing Radian Measure

A central angle, θ, in a circle of radius 6 inches intercepts an arc of length 15 inches. What is the radian measure of θ?

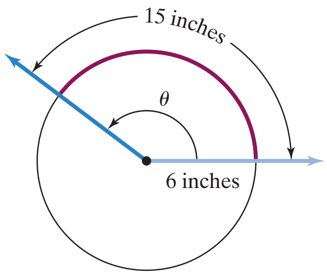


Figure 4.8

Solution Angle θ is shown in **Figure 4.8**. The radian measure of a central angle is the length of the intercepted arc, s, divided by the circle's radius, r. The length of the intercepted arc is 15 inches: $s = 15$ inches. The circle's radius is 6 inches: $r = 6$ inches. Now we use the formula for radian measure to find the radian measure of θ.

$$\theta = \frac{s}{r} = \frac{15 \text{ inches}}{6 \text{ inches}} = 2.5$$

Thus, the radian measure of θ is 2.5.

Study Tip

Before applying the formula for radian measure, be sure that the same unit of length is used for the intercepted arc, s, and the radius, r.

In Example 1, notice that the units (inches) cancel when we use the formula for radian measure. We are left with a number with no units. Thus, if an angle θ has a measure of 2.5 radians, we can write $\theta = 2.5$ radians or $\theta = 2.5$. We will often include the word *radians* simply for emphasis. There should be no confusion as to whether radian or degree measure is being used. Why is this so? If θ has a degree measure of, say, 2.5°, we must include the degree symbol and write $\theta = 2.5°$, and *not* $\theta = 2.5$.

Check Point 1 A central angle, θ, in a circle of radius 12 feet intercepts an arc of length 42 feet. What is the radian measure of θ?

4 Convert between degrees and radians.

Relationship between Degrees and Radians

How can we obtain a relationship between degrees and radians? We compare the number of degrees and the number of radians in one complete rotation, shown in **Figure 4.9**. We know that 360° is the amount of rotation of a ray back onto itself. The length of the intercepted arc is equal to the circumference of the circle. Thus, the radian measure of this central angle is the circumference of the circle divided by the circle's radius, r. The circumference of a circle of radius r is $2\pi r$. We use the formula for radian measure to find the radian measure of the 360° angle.

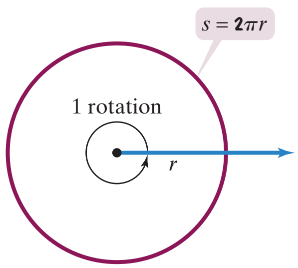


Figure 4.9 A complete rotation

$$\theta = \frac{s}{r} = \frac{\text{the circle's circumference}}{r} = \frac{2\pi r}{r} = 2\pi$$

Because one complete rotation measures 360° and 2π radians,

$$360° = 2\pi \text{ radians.}$$

Dividing both sides by 2, we have

$$180° = \pi \text{ radians.}$$

Dividing this last equation by 180° or π gives the conversion rules in the box on the next page.

Study Tip

The unit you are converting *to* appears in the *numerator* of the conversion factor.

Conversion between Degrees and Radians

Using the basic relationship π radians $= 180°$,

1. To convert degrees to radians, multiply degrees by $\dfrac{\pi \text{ radians}}{180°}$.
2. To convert radians to degrees, multiply radians by $\dfrac{180°}{\pi \text{ radians}}$.

Angles that are fractions of a complete rotation are usually expressed in radian measure as fractional multiples of π, rather than as decimal approximations. For example, we write $\theta = \frac{\pi}{2}$ rather than using the decimal approximation $\theta \approx 1.57$.

EXAMPLE 2 Converting from Degrees to Radians

Convert each angle in degrees to radians:

a. 30° **b.** 90° **c.** −135°.

Solution To convert degrees to radians, multiply by $\dfrac{\pi \text{ radians}}{180°}$. Observe how the degree units cancel.

a. $30° = 30° \cdot \dfrac{\pi \text{ radians}}{180°} = \dfrac{30\pi}{180} \text{ radians} = \dfrac{\pi}{6} \text{ radians}$

b. $90° = 90° \cdot \dfrac{\pi \text{ radians}}{180°} = \dfrac{90\pi}{180} \text{ radians} = \dfrac{\pi}{2} \text{ radians}$

c. $-135° = -135° \cdot \dfrac{\pi \text{ radians}}{180°} = -\dfrac{135\pi}{180} \text{ radians} = -\dfrac{3\pi}{4} \text{ radians}$

Divide the numerator and denominator by 45.

Check Point 2 Convert each angle in degrees to radians:

a. 60° **b.** 270° **c.** −300°.

EXAMPLE 3 Converting from Radians to Degrees

Convert each angle in radians to degrees:

a. $\dfrac{\pi}{3}$ radians **b.** $-\dfrac{5\pi}{3}$ radians **c.** 1 radian.

Solution To convert radians to degrees, multiply by $\dfrac{180°}{\pi \text{ radians}}$. Observe how the radian units cancel.

a. $\dfrac{\pi}{3} \text{ radians} = \dfrac{\pi \text{ radians}}{3} \cdot \dfrac{180°}{\pi \text{ radians}} = \dfrac{180°}{3} = 60°$

b. $-\dfrac{5\pi}{3} \text{ radians} = -\dfrac{5\pi \text{ radians}}{3} \cdot \dfrac{180°}{\pi \text{ radians}} = -\dfrac{5 \cdot 180°}{3} = -300°$

c. $1 \text{ radian} = 1 \text{ radian} \cdot \dfrac{180°}{\pi \text{ radians}} = \dfrac{180°}{\pi} \approx 57.3°$

Study Tip

In Example 3(c), we see that 1 radian is approximately 57°. Keep in mind that a radian is much larger than a degree.

Check Point 3 Convert each angle in radians to degrees:

a. $\dfrac{\pi}{4}$ radians **b.** $-\dfrac{4\pi}{3}$ radians **c.** 6 radians.

5 Draw angles in standard position.

Drawing Angles in Standard Position

Although we can convert angles in radians to degrees, it is helpful to "think in radians" without having to make this conversion. To become comfortable with radian measure, consider angles in standard position: Each vertex is at the origin and each initial side lies along the positive x-axis. Think of the terminal side of the angle revolving around the origin. Thinking in radians means determining what part of a complete revolution or how many full revolutions will produce an angle whose radian measure is known. And here's the thing: We want to do this without having to convert from radians to degrees.

Figure 4.10 is a starting point for learning to think in radians. The figure illustrates that when the terminal side makes one full revolution, it forms an angle whose radian measure is 2π. The figure shows the quadrantal angles formed by $\frac{3}{4}$ of a revolution, $\frac{1}{2}$ of a revolution, and $\frac{1}{4}$ of a revolution.

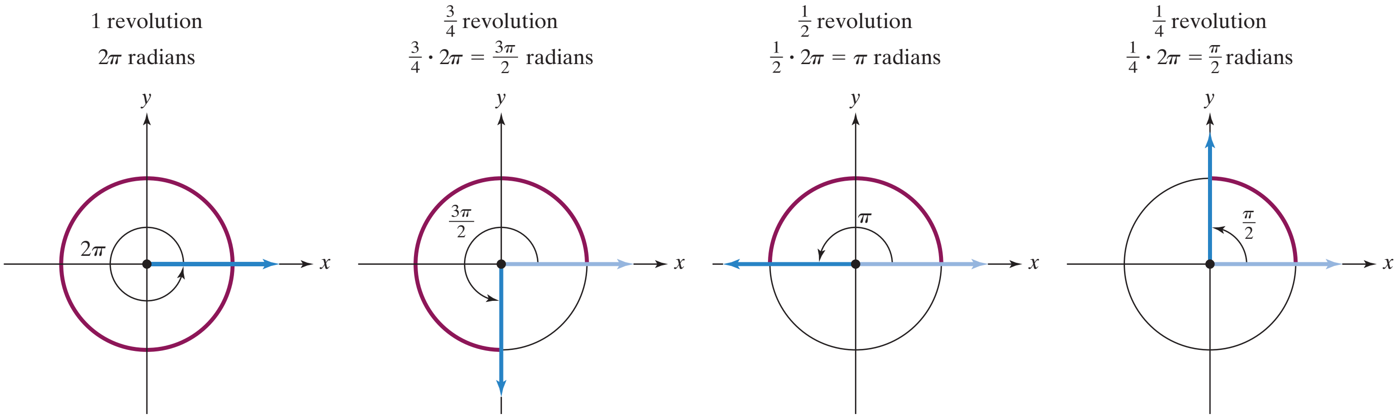


Figure 4.10 Angles formed by revolutions of terminal sides

EXAMPLE 4 Drawing Angles in Standard Position

Draw and label each angle in standard position:

a. $\theta = \frac{\pi}{4}$ (theta) **b.** $\alpha = \frac{5\pi}{4}$ (alpha) **c.** $\beta = -\frac{3\pi}{4}$ (beta) **d.** $\gamma = \frac{9\pi}{4}$. (gamma)

Solution Because we are drawing angles in standard position, each vertex is at the origin and each initial side lies along the positive x-axis.

a. An angle of $\frac{\pi}{4}$ radians is a positive angle. It is obtained by rotating the terminal side counterclockwise. Because 2π is a full-circle revolution, we can express $\frac{\pi}{4}$ as a fractional part of 2π to determine the necessary rotation:

$$\frac{\pi}{4} = \frac{1}{8} \cdot 2\pi.$$

$\frac{\pi}{4}$ is $\frac{1}{8}$ of a complete revolution of 2π radians.

We see that $\theta = \frac{\pi}{4}$ is obtained by rotating the terminal side counterclockwise for $\frac{1}{8}$ of a revolution. The angle lies in quadrant I and is shown in **Figure 4.11**.

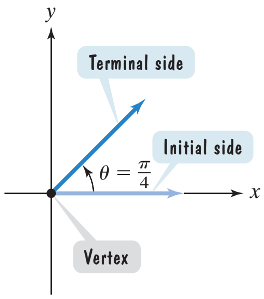


Figure 4.11

b. An angle of $\frac{5\pi}{4}$ radians is a positive angle. It is obtained by rotating the terminal side counterclockwise. Here are two ways to determine the necessary rotation:

Method 1

$$\frac{5\pi}{4} = \frac{5}{8} \cdot 2\pi$$

$\frac{5\pi}{4}$ is $\frac{5}{8}$ of a complete revolution of 2π radians.

Method 2

$$\frac{5\pi}{4} = \pi + \frac{\pi}{4}.$$

π is a half-circle revolution.

$\frac{\pi}{4}$ is $\frac{1}{8}$ of a complete revolution.

Method 1 shows that $\alpha = \frac{5\pi}{4}$ is obtained by rotating the terminal side counterclockwise for $\frac{5}{8}$ of a revolution. Method 2 shows that $\alpha = \frac{5\pi}{4}$ is obtained by rotating the terminal side counterclockwise for half of a revolution followed by a counterclockwise rotation of $\frac{1}{8}$ of a revolution. The angle lies in quadrant III and is shown in **Figure 4.12**.

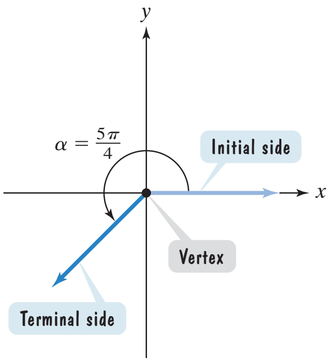


Figure 4.12

c. An angle of $-\frac{3\pi}{4}$ is a negative angle. It is obtained by rotating the terminal side clockwise. We use $\left|-\frac{3\pi}{4}\right|$, or $\frac{3\pi}{4}$, to determine the necessary rotation.

Method 1

$$\frac{3\pi}{4} = \frac{3}{8} \cdot 2\pi$$

$\frac{3\pi}{4}$ is $\frac{3}{8}$ of a complete revolution of 2π radians.

Method 2

$$\frac{3\pi}{4} = \frac{2\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2} + \frac{\pi}{4}$$

$\frac{\pi}{2}$ is a quarter-circle revolution.

$\frac{\pi}{4}$ is $\frac{1}{8}$ of a complete revolution.

Method 1 shows that $\beta = -\frac{3\pi}{4}$ is obtained by rotating the terminal side clockwise for $\frac{3}{8}$ of a revolution. Method 2 shows that $\beta = -\frac{3\pi}{4}$ is obtained by rotating the terminal side clockwise for $\frac{1}{4}$ of a revolution followed by a clockwise rotation of $\frac{1}{8}$ of a revolution. The angle lies in quadrant III and is shown in **Figure 4.13**.

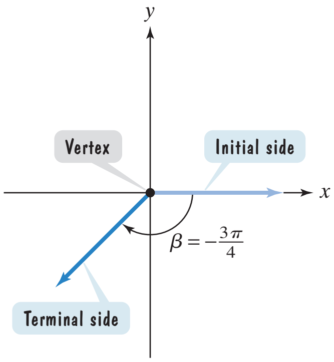


Figure 4.13

d. An angle of $\frac{9\pi}{4}$ radians is a positive angle. It is obtained by rotating the terminal side counterclockwise. Here are two methods to determine the necessary rotation:

Method 1

$$\frac{9\pi}{4} = \frac{9}{8} \cdot 2\pi$$

$\frac{9\pi}{4}$ is $\frac{9}{8}$, or $1\frac{1}{8}$, complete revolutions of 2π radians.

Method 2

$$\frac{9\pi}{4} = 2\pi + \frac{\pi}{4}.$$

2π is a full-circle revolution.

$\frac{\pi}{4}$ is $\frac{1}{8}$ of a complete revolution.

Method 1 shows that $\gamma = \frac{9\pi}{4}$ is obtained by rotating the terminal side counterclockwise for $1\frac{1}{8}$ revolutions. Method 2 shows that $\gamma = \frac{9\pi}{4}$ is obtained by rotating the terminal side counterclockwise for a full-circle revolution followed by a counterclockwise rotation of $\frac{1}{8}$ of a revolution. The angle lies in quadrant I and is shown in **Figure 4.14**.

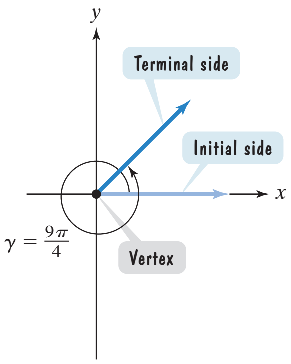


Figure 4.14

Check Point 4 Draw and label each angle in standard position:

a. $\theta = -\frac{\pi}{4}$ **b.** $\alpha = \frac{3\pi}{4}$ **c.** $\beta = -\frac{7\pi}{4}$ **d.** $\gamma = \frac{13\pi}{4}$.

Figure 4.15 illustrates the degree and radian measures of angles that you will commonly see in trigonometry. Each angle is in standard position, so that the initial side lies along the positive x-axis. We will be using both degree and radian measure for these angles.

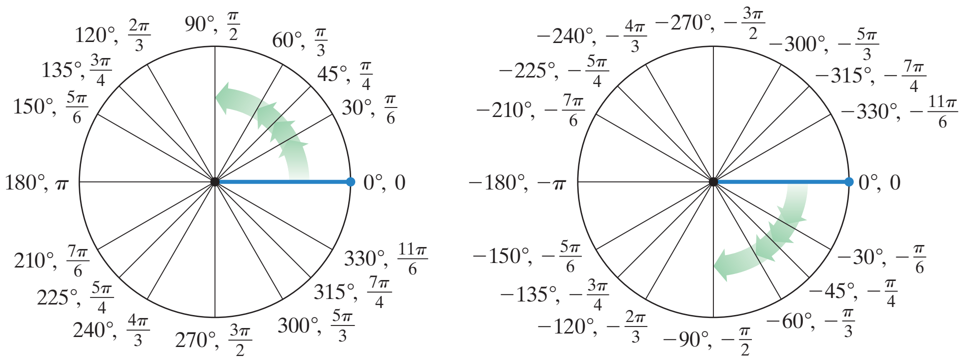
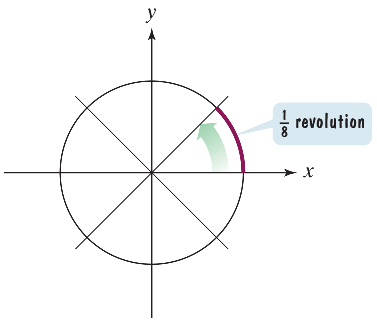


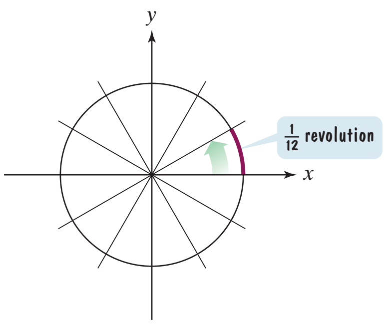
Figure 4.15 Degree and radian measures of selected positive and negative angles

Study Tip

When drawing the angles in **Table 4.1** and **Figure 4.15**, it is helpful to first divide the rectangular coordinate system into eight equal sectors:



or 12 equal sectors:



Perhaps we should call this study tip "Making a Clone of Arc."

Table 4.1 describes some of the positive angles in **Figure 4.15** in terms of revolutions of the angle's terminal side around the origin.

Table 4.1

Terminal Side	Radian Measure of Angle	Degree Measure of Angle
$\frac{1}{12}$ revolution	$\frac{1}{12} \cdot 2\pi = \frac{\pi}{6}$	$\frac{1}{12} \cdot 360° = 30°$
$\frac{1}{8}$ revolution	$\frac{1}{8} \cdot 2\pi = \frac{\pi}{4}$	$\frac{1}{8} \cdot 360° = 45°$
$\frac{1}{6}$ revolution	$\frac{1}{6} \cdot 2\pi = \frac{\pi}{3}$	$\frac{1}{6} \cdot 360° = 60°$
$\frac{1}{4}$ revolution	$\frac{1}{4} \cdot 2\pi = \frac{\pi}{2}$	$\frac{1}{4} \cdot 360° = 90°$
$\frac{1}{3}$ revolution	$\frac{1}{3} \cdot 2\pi = \frac{2\pi}{3}$	$\frac{1}{3} \cdot 360° = 120°$
$\frac{1}{2}$ revolution	$\frac{1}{2} \cdot 2\pi = \pi$	$\frac{1}{2} \cdot 360° = 180°$
$\frac{2}{3}$ revolution	$\frac{2}{3} \cdot 2\pi = \frac{4\pi}{3}$	$\frac{2}{3} \cdot 360° = 240°$
$\frac{3}{4}$ revolution	$\frac{3}{4} \cdot 2\pi = \frac{3\pi}{2}$	$\frac{3}{4} \cdot 360° = 270°$
$\frac{7}{8}$ revolution	$\frac{7}{8} \cdot 2\pi = \frac{7\pi}{4}$	$\frac{7}{8} \cdot 360° = 315°$
1 revolution	$1 \cdot 2\pi = 2\pi$	$1 \cdot 360° = 360°$

6 Find coterminal angles.

Coterminal Angles

Two angles with the same initial and terminal sides but possibly different rotations are called **coterminal angles**.

Every angle has infinitely many coterminal angles. Why? Think of an angle in standard position. If the rotation of the angle is extended by one or more complete rotations of 360° or 2π, clockwise or counterclockwise, the result is an angle with the same initial and terminal sides as the original angle.

Coterminal Angles

Increasing or decreasing the degree measure of an angle in standard position by an integer multiple of 360° results in a coterminal angle. Thus, an angle of $\theta°$ is coterminal with angles of $\theta° \pm 360°k$, where k is an integer.

Increasing or decreasing the radian measure of an angle by an integer multiple of 2π results in a coterminal angle. Thus, an angle of θ radians is coterminal with angles of $\theta \pm 2\pi k$, where k is an integer.

Two coterminal angles for an angle of $\theta°$ can be found by adding 360° to $\theta°$ and subtracting 360° from $\theta°$.

EXAMPLE 5 Finding Coterminal Angles

Assume the following angles are in standard position. Find a positive angle less than 360° that is coterminal with each of the following:

a. a 420° angle **b.** a −120° angle.

Solution We obtain the coterminal angle by adding or subtracting 360°. The requirement to obtain a positive angle less than 360° determines whether we should add or subtract.

a. For a 420° angle, subtract 360° to find a positive coterminal angle.

$$420° - 360° = 60°$$

A 60° angle is coterminal with a 420° angle. **Figure 4.16(a)** illustrates that these angles have the same initial and terminal sides.

b. For a −120° angle, add 360° to find a positive coterminal angle.

$$-120° + 360° = 240°$$

A 240° angle is coterminal with a −120° angle. **Figure 4.16(b)** illustrates that these angles have the same initial and terminal sides.

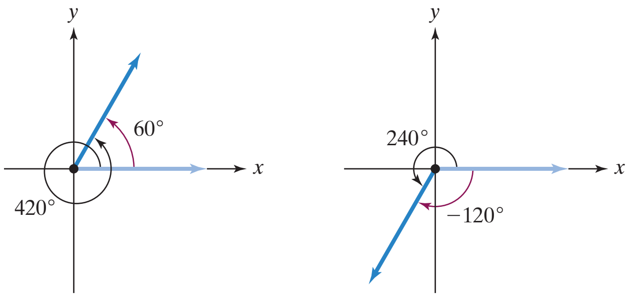


Figure 4.16 Pairs of coterminal angles

(a) Angles of 420° and 60° are coterminal.

(b) Angles of −120° and 240° are coterminal.

Check Point 5 Find a positive angle less than 360° that is coterminal with each of the following:

a. a 400° angle **b.** a −135° angle.

Two coterminal angles for an angle of θ radians can be found by adding 2π to θ and subtracting 2π from θ.

EXAMPLE 6 Finding Coterminal Angles

Assume the following angles are in standard position. Find a positive angle less than 2π that is coterminal with each of the following:

a. a $\frac{17\pi}{6}$ angle **b.** a $-\frac{\pi}{12}$ angle.

Solution We obtain the coterminal angle by adding or subtracting 2π. The requirement to obtain a positive angle less than 2π determines whether we should add or subtract.

a. For a $\frac{17\pi}{6}$, or $2\frac{5}{6}\pi$, angle, subtract 2π to find a positive coterminal angle.

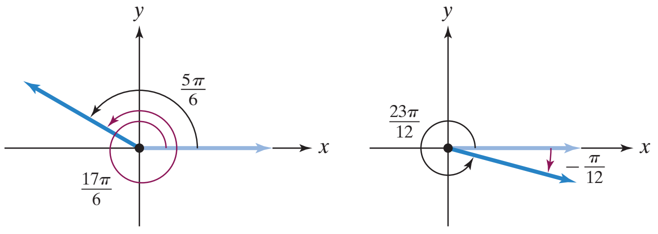
$$\frac{17\pi}{6} - 2\pi = \frac{17\pi}{6} - \frac{12\pi}{6} = \frac{5\pi}{6}$$

A $\frac{5\pi}{6}$ angle is coterminal with a $\frac{17\pi}{6}$ angle. **Figure 4.17(a)** illustrates that these angles have the same initial and terminal sides.

b. For a $-\frac{\pi}{12}$ angle, add 2π to find a positive coterminal angle.

$$-\frac{\pi}{12} + 2\pi = -\frac{\pi}{12} + \frac{24\pi}{12} = \frac{23\pi}{12}$$

A $\frac{23\pi}{12}$ angle is coterminal with a $-\frac{\pi}{12}$ angle. **Figure 4.17(b)** illustrates that these angles have the same initial and terminal sides.



(a) Angles of $\frac{17\pi}{6}$ and $\frac{5\pi}{6}$ are coterminal.

(b) Angles of $-\frac{\pi}{12}$ and $\frac{23\pi}{12}$ are coterminal.

Figure 4.17 Pairs of coterminal angles

Check Point 6 Find a positive angle less than 2π that is coterminal with each of the following:

a. a $\frac{13\pi}{5}$ angle **b.** a $-\frac{\pi}{15}$ angle.

To find a positive coterminal angle less than 360° or 2π, it is sometimes necessary to add or subtract more than one multiple of 360° or 2π.

EXAMPLE 7 Finding Coterminal Angles

Find a positive angle less than 360° or 2π that is coterminal with each of the following:

a. a 750° angle **b.** a $\frac{22\pi}{3}$ angle **c.** a $-\frac{17\pi}{6}$ angle.

Solution

a. For a 750° angle, subtract two multiples of 360°, or 720°, to find a positive coterminal angle less than 360°.

$$750° - 360° \cdot 2 = 750° - 720° = 30°$$

A 30° angle is coterminal with a 750° angle.

b. For a $\frac{22\pi}{3}$, or $7\frac{1}{3}\pi$, angle, subtract three multiples of 2π, or 6π, to find a positive coterminal angle less than 2π.

$$\frac{22\pi}{3} - 2\pi \cdot 3 = \frac{22\pi}{3} - 6\pi = \frac{22\pi}{3} - \frac{18\pi}{3} = \frac{4\pi}{3}$$

A $\frac{4\pi}{3}$ angle is coterminal with a $\frac{22\pi}{3}$ angle.

c. For a $-\frac{17\pi}{6}$, or $-2\frac{5}{6}\pi$ angle, add two multiples of 2π, or 4π, to find a positive coterminal angle less than 2π.

$$-\frac{17\pi}{6} + 2\pi \cdot 2 = -\frac{17\pi}{6} + 4\pi = -\frac{17\pi}{6} + \frac{24\pi}{6} = \frac{7\pi}{6}$$

A $\frac{7\pi}{6}$ angle is coterminal with a $-\frac{17\pi}{6}$ angle.

Discovery

Make a sketch for each part of Example 7 illustrating that the coterminal angle we found and the given angle have the same initial and terminal sides.

Check Point 7 Find a positive angle less than 360° or 2π that is coterminal with each of the following:

a. an 855° angle **b.** a $\frac{17\pi}{3}$ angle **c.** a $-\frac{25\pi}{6}$ angle.

7 Find the length of a circular arc.

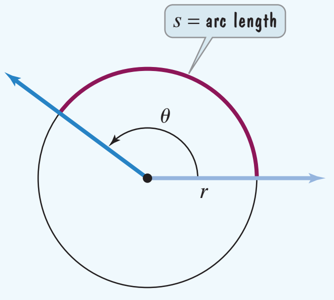
The Length of a Circular Arc

We can use the radian measure formula, $\theta = \frac{s}{r}$, to find the length of the arc of a circle. How do we do this? Remember that s represents the length of the arc intercepted by the central angle θ. Thus, by solving the formula for s, we have an equation for arc length.

The Length of a Circular Arc

Let r be the radius of a circle and θ the nonnegative radian measure of a central angle of the circle. The length of the arc intercepted by the central angle is

$$s = r\theta.$$



EXAMPLE 8 **Finding the Length of a Circular Arc**

A circle has a radius of 10 inches. Find the length of the arc intercepted by a central angle of 120°.

Solution The formula $s = r\theta$ can be used only when θ is expressed in radians. Thus, we begin by converting 120° to radians. Multiply by $\frac{\pi \text{ radians}}{180°}$.

$$120° = 120° \cdot \frac{\pi \text{ radians}}{180°} = \frac{120\pi}{180} \text{ radians} = \frac{2\pi}{3} \text{ radians}$$

Now we can use the formula $s = r\theta$ to find the length of the arc. The circle's radius is 10 inches: $r = 10$ inches. The measure of the central angle, in radians, is $\frac{2\pi}{3}$: $\theta = \frac{2\pi}{3}$. The length of the arc intercepted by this central angle is

$$s = r\theta = (10 \text{ inches})\left(\frac{2\pi}{3}\right) = \frac{20\pi}{3} \text{ inches} \approx 20.94 \text{ inches.}$$

Study Tip

The unit used to describe the length of a circular arc is the same unit that is given in the circle's radius.

Check Point 8 A circle has a radius of 6 inches. Find the length of the arc intercepted by a central angle of 45°. Express arc length in terms of π. Then round your answer to two decimal places.

8 Use linear and angular speed to describe motion on a circular path.

Linear and Angular Speed

A carousel contains four circular rows of animals. As the carousel revolves, the animals in the outer row travel a greater distance per unit of time than those in the inner rows. These animals have a greater *linear speed* than those in the inner rows. By contrast, all animals, regardless of the row, complete the same number of revolutions per unit of time. All animals in the four circular rows travel at the same *angular speed*.

Using v for linear speed and ω (omega) for angular speed, we define these two kinds of speeds along a circular path as follows:

Definitions of Linear and Angular Speed

If a point is in motion on a circle of radius r through an angle of θ radians in time t, then its **linear speed** is

$$v = \frac{s}{t},$$

where s is the arc length given by $s = r\theta$, and its **angular speed** is

$$\omega = \frac{\theta}{t}.$$

The hard drive in a computer rotates at 3600 revolutions per minute. This angular speed, expressed in revolutions per minute, can also be expressed in revolutions per second, radians per minute, and radians per second. Using 2π radians $= 1$ revolution, we express the angular speed of a hard drive in radians per minute as follows:

$$3600 \text{ revolutions per minute}$$
$$= \frac{3600 \text{ revolutions}}{1 \text{ minute}} \cdot \frac{2\pi \text{ radians}}{1 \text{ revolution}} = \frac{7200\pi \text{ radians}}{1 \text{ minute}}$$
$$= 7200\pi \text{ radians per minute.}$$

We can establish a relationship between the two kinds of speed by dividing both sides of the arc length formula, $s = r\theta$, by t:

$$\frac{s}{t} = \frac{r\theta}{t} = r\frac{\theta}{t}.$$

This expression defines linear speed. This expression defines angular speed.

Thus, linear speed is the product of the radius and the angular speed.

Linear Speed in Terms of Angular Speed

The linear speed, v, of a point a distance r from the center of rotation is given by

$$v = r\omega,$$

where ω is the angular speed in radians per unit of time.

EXAMPLE 9 Finding Linear Speed

A wind machine used to generate electricity has blades that are 10 feet in length (see **Figure 4.18**). The propeller is rotating at four revolutions per second. Find the linear speed, in feet per second, of the tips of the blades.

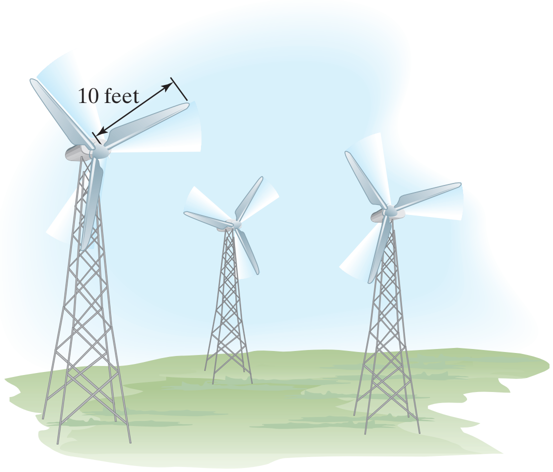


Figure 4.18

Solution We are given ω, the angular speed.

$$\omega = 4 \text{ revolutions per second}$$

We use the formula $v = r\omega$ to find v, the linear speed. Before applying the formula, we must express ω in radians per second.

$$\omega = \frac{4 \text{ revolutions}}{1 \text{ second}} \cdot \frac{2\pi \text{ radians}}{1 \text{ revolution}} = \frac{8\pi \text{ radians}}{1 \text{ second}} \text{ or } \frac{8\pi}{1 \text{ second}}$$

The angular speed of the propeller is 8π radians per second. The linear speed is

$$v = r\omega = 10 \text{ feet} \cdot \frac{8\pi}{1 \text{ second}} = \frac{80\pi \text{ feet}}{\text{second}}.$$

The linear speed of the tips of the blades is 80π feet per second, which is approximately 251 feet per second.

Check Point 9 Long before iPods that hold thousands of songs and play them with superb audio quality, individual songs were delivered on 75-rpm and 45-rpm circular records. A 45-rpm record has an angular speed of 45 revolutions per minute. Find the linear speed, in inches per minute, at the point where the needle is 1.5 inches from the record's center.

Exercise Set 4.1

Practice Exercises

In Exercises 1–6, the measure of an angle is given. Classify the angle as acute, right, obtuse, or straight.

1. 135° **2.** 177°

3. 83.135° **4.** 87.177°

5. π **6.** $\frac{\pi}{2}$

In Exercises 7–12, find the radian measure of the central angle of a circle of radius r that intercepts an arc of length s.

Radius, r	Arc Length, s
7. 10 inches	40 inches
8. 5 feet	30 feet
9. 6 yards	8 yards
10. 8 yards	18 yards
11. 1 meter	400 centimeters
12. 1 meter	600 centimeters

In Exercises 13–20, convert each angle in degrees to radians. Express your answer as a multiple of π.

13. 45° **14.** 18° **15.** 135°

16. 150° **17.** 300° **18.** 330°

19. −225° **20.** −270°

In Exercises 21–28, convert each angle in radians to degrees.

21. $\frac{\pi}{2}$ **22.** $\frac{\pi}{9}$ **23.** $\frac{2\pi}{3}$

24. $\frac{3\pi}{4}$ **25.** $\frac{7\pi}{6}$ **26.** $\frac{11\pi}{6}$

27. -3π **28.** -4π

In Exercises 29–34, convert each angle in degrees to radians. Round to two decimal places.

29. 18° **30.** 76° **31.** −40°

32. −50° **33.** 200° **34.** 250°

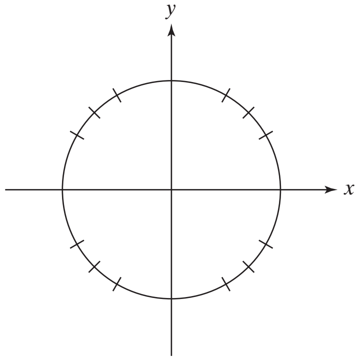
In Exercises 35–40, convert each angle in radians to degrees. Round to two decimal places.

35. 2 radians **36.** 3 radians

37. $\frac{\pi}{13}$ radians **38.** $\frac{\pi}{17}$ radians

39. −4.8 radians **40.** −5.2 radians

In Exercises 41–56, use the circle shown in the rectangular coordinate system to draw each angle in standard position. State the quadrant in which the angle lies. When an angle's measure is given in radians, work the exercise without converting to degrees.



41. $\frac{7\pi}{6}$ **42.** $\frac{4\pi}{3}$ **43.** $\frac{3\pi}{4}$

44. $\frac{7\pi}{4}$ **45.** $-\frac{2\pi}{3}$ **46.** $-\frac{5\pi}{6}$

47. $-\frac{5\pi}{4}$ **48.** $-\frac{7\pi}{4}$ **49.** $\frac{16\pi}{3}$

50. $\frac{14\pi}{3}$ **51.** 120° **52.** 150°

53. −210° **54.** −240° **55.** 420°

56. 405°

In Exercises 57–70, find a positive angle less than 360° or 2π that is coterminal with the given angle.

57. 395° **58.** 415° **59.** −150°

60. −160° **61.** −765° **62.** −760°

63. $\frac{19\pi}{6}$ **64.** $\frac{17\pi}{5}$ **65.** $\frac{23\pi}{5}$

66. $\frac{25\pi}{6}$ **67.** $-\frac{\pi}{50}$ **68.** $-\frac{\pi}{40}$

69. $-\frac{31\pi}{7}$ **70.** $-\frac{38\pi}{9}$

In Exercises 71–74, find the length of the arc on a circle of radius r intercepted by a central angle θ. Express arc length in terms of π. Then round your answer to two decimal places.

Radius, *r*	Central Angle, θ
71. 12 inches	$\theta = 45°$
72. 16 inches	$\theta = 60°$
73. 8 feet	$\theta = 225°$
74. 9 yards	$\theta = 315°$

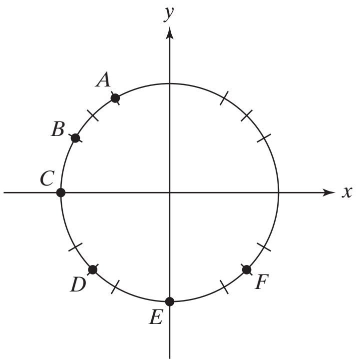
In Exercises 75–76, express each angular speed in radians per second.

75. 6 revolutions per second

76. 20 revolutions per second

Practice Plus

Use the circle shown in the rectangular coordinate system to solve Exercises 77–82. Find two angles, in radians, between -2π and 2π such that each angle's terminal side passes through the origin and the given point.



77. *A* **78.** *B*

79. *D* **80.** *F*

81. *E* **82.** *C*

In Exercises 83–86, find the positive radian measure of the angle that the second hand of a clock moves through in the given time.

83. 55 seconds **84.** 35 seconds

85. 3 minutes and 40 seconds

86. 4 minutes and 25 seconds

Application Exercises

87. The minute hand of a clock moves from 12 to 2 o'clock, or $\frac{1}{6}$ of a complete revolution. Through how many degrees does it move? Through how many radians does it move?

88. The minute hand of a clock moves from 12 to 4 o'clock, or $\frac{1}{3}$ of a complete revolution. Through how many degrees does it move? Through how many radians does it move?

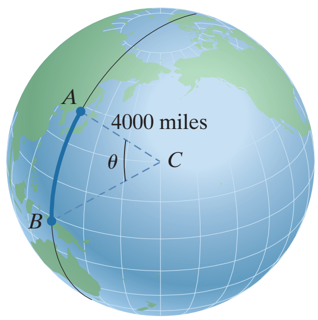
89. The minute hand of a clock is 8 inches long and moves from 12 to 2 o'clock. How far does the tip of the minute hand move? Express your answer in terms of π and then round to two decimal places.

90. The minute hand of a clock is 6 inches long and moves from 12 to 4 o'clock. How far does the tip of the minute hand move? Express your answer in terms of π and then round to two decimal places.

91. The figure shows a highway sign that warns of a railway crossing. The lines that form the cross pass through the circle's center and intersect at right angles. If the radius of the circle is 24 inches, find the length of each of the four arcs formed by the cross. Express your answer in terms of π and then round to two decimal places.

92. The radius of a wheel rolling on the ground is 80 centimeters. If the wheel rotates through an angle of 60°, how many centimeters does it move? Express your answer in terms of π and then round to two decimal places.

How do we measure the distance between two points, A and B, on Earth? We measure along a circle with a center, C, at the center of Earth. The radius of the circle is equal to the distance from C to the surface. Use the fact that Earth is a sphere of radius equal to approximately 4000 miles to solve Exercises 93–96.



93. If two points, A and B, are 8000 miles apart, express angle θ in radians and in degrees.

94. If two points, A and B, are 10,000 miles apart, express angle θ in radians and in degrees.

95. If $\theta = 30°$, find the distance between A and B to the nearest mile.

96. If $\theta = 10°$, find the distance between A and B to the nearest mile.

97. The angular speed of a point on Earth is $\frac{\pi}{12}$ radian per hour. The Equator lies on a circle of radius approximately 4000 miles. Find the linear velocity, in miles per hour, of a point on the Equator.

98. A Ferris wheel has a radius of 25 feet. The wheel is rotating at two revolutions per minute. Find the linear speed, in feet per minute, of a seat on this Ferris wheel.

99. A water wheel has a radius of 12 feet. The wheel is rotating at 20 revolutions per minute. Find the linear speed, in feet per minute, of the water.

100. On a carousel, the outer row of animals is 20 feet from the center. The inner row of animals is 10 feet from the center. The carousel is rotating at 2.5 revolutions per minute. What is the difference, in feet per minute, in the linear speeds of the animals in the outer and inner rows? Round to the nearest foot per minute.

Writing in Mathematics

101. What is an angle?

102. What determines the size of an angle?

103. Describe an angle in standard position.

104. Explain the difference between positive and negative angles. What are coterminal angles?

105. Explain what is meant by one radian.

106. Explain how to find the radian measure of a central angle.

107. Describe how to convert an angle in degrees to radians.

108. Explain how to convert an angle in radians to degrees.

109. Explain how to find the length of a circular arc.

110. If a carousel is rotating at 2.5 revolutions per minute, explain how to find the linear speed of a child seated on one of the animals.

111. The angular velocity of a point on Earth is $\frac{\pi}{12}$ radian per hour. Describe what happens every 24 hours.

112. Have you ever noticed that we use the vocabulary of angles in everyday speech? Here is an example:

> My opinion about art museums took a 180° turn after visiting the San Francisco Museum of Modern Art.

Explain what this means. Then give another example of the vocabulary of angles in everyday use.

Technology Exercises

In Exercises 113–116, use the keys on your calculator or graphing utility for converting an angle in degrees, minutes, and seconds ($D°M'S''$) into decimal form, and vice versa.

In Exercises 113–114, convert each angle to a decimal in degrees. Round your answer to two decimal places.

113. 30°15′10″ **114.** 65°45′20″

In Exercises 115–116, convert each angle to $D°M'S''$ form. Round your answer to the nearest second.

115. 30.42° **116.** 50.42°

Critical Thinking Exercises

Make Sense? *In Exercises 117–120, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

117. I made an error because the angle I drew in standard position exceeded a straight angle.

118. When an angle's measure is given in terms of π, I know that it's measured using radians.

119. When I convert degrees to radians, I multiply by 1, choosing $\frac{\pi}{180°}$ for 1.

120. Using radian measure, I can always find a positive angle less than 2π coterminal with a given angle by adding or subtracting 2π.

121. If $\theta = \frac{3}{2}$, is this angle larger or smaller than a right angle?

122. A railroad curve is laid out on a circle. What radius should be used if the track is to change direction by 20° in a distance of 100 miles? Round your answer to the nearest mile.

123. Assuming Earth to be a sphere of radius 4000 miles, how many miles north of the Equator is Miami, Florida, if it is 26° north from the Equator? Round your answer to the nearest mile.

Preview Exercises

Exercises 124–126 will help you prepare for the material covered in the next section.

124. Graph: $x^2 + y^2 = 1$. Then locate the point $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ on the graph.

125. Use your graph of $x^2 + y^2 = 1$ from Exercise 124 to determine the relation's domain and range.

126. Find $\frac{x}{y}$ for $x = -\frac{1}{2}$ and $y = \frac{\sqrt{3}}{2}$, and then rationalize the denominator.

Section 4.2 Trigonometric Functions: The Unit Circle

Objectives

1. Use a unit circle to define trigonometric functions of real numbers.
2. Recognize the domain and range of sine and cosine functions.
3. Find exact values of the trigonometric functions at $\frac{\pi}{4}$.
4. Use even and odd trigonometric functions.
5. Recognize and use fundamental identities.
6. Use periodic properties.
7. Evaluate trigonometric functions with a calculator.

There is something comforting in the repetition of some of nature's patterns. The ocean level at a beach varies between high and low tide approximately every 12 hours. The number of hours of daylight oscillates from a maximum on the summer solstice, June 21, to a minimum on the winter solstice, December 21. Then it increases to the same maximum the following June 21. Some believe that cycles, called biorhythms, represent physical, emotional, and intellectual aspects of our lives. In this chapter, we study six functions, the six *trigonometric functions*, that are used to model phenomena that occur again and again.

Calculus and the Unit Circle

The word *trigonometry* means measurement of triangles. Trigonometric functions, with domains consisting of sets of angles, were first defined using right triangles. By contrast, problems in calculus are solved using functions whose domains are sets of real numbers. Therefore, we introduce the trigonometric functions using unit circles and radians, rather than right triangles and degrees.

A **unit circle** is a circle of radius 1, with its center at the origin of a rectangular coordinate system. The equation of this unit circle is $x^2 + y^2 = 1$. **Figure 4.19** shows a unit circle with a central angle measuring t radians.

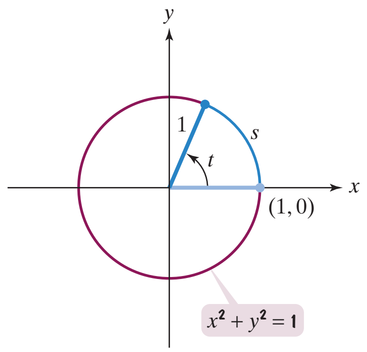


Figure 4.19 Unit circle with a central angle measuring t radians

We can use the formula for the length of a circular arc, $s = r\theta$, to find the length of the intercepted arc.

$$s = r\theta = 1 \cdot t = t$$

The radius of a unit circle is 1.

The radian measure of the central angle is t.

Thus, the length of the intercepted arc is t. This is also the radian measure of the central angle. Thus, **in a unit circle, the radian measure of the central angle is equal to the length of the intercepted arc**. Both are given by the same *real number* t.