140. If $\log(7x + 3) - \log(2x + 5) = 4$, then the equation in exponential form is $10^4 = (7x + 3) - (2x + 5)$.

141. If $x = \frac{1}{k}\ln y$, then $y = e^{kx}$.

142. Examples of exponential equations include $10^x = 5.71$, $e^x = 0.72$, and $x^{10} = 5.71$.

143. If \$4000 is deposited into an account paying 3% interest compounded annually and at the same time \$2000 is deposited into an account paying 5% interest compounded annually, after how long will the two accounts have the same balance? Round to the nearest year.

Solve each equation in Exercises 144–146. Check each proposed solution by direct substitution or with a graphing utility.

144. $(\ln x)^2 = \ln x^2$

145. $(\log x)(2 \log x + 1) = 6$

146. $\ln(\ln x) = 0$

Group Exercise

147. Research applications of logarithmic functions as mathematical models and plan a seminar based on your group's research. Each group member should research one of the following areas or any other area of interest: pH (acidity of solutions), intensity of sound (decibels), brightness of stars, human memory, progress over time in a sport, profit over time. For the area that you select, explain how logarithmic functions are used and provide examples.

Preview Exercises

Exercises 148–150 will help you prepare for the material covered in the next section.

148. The formula $A = 10e^{-0.003t}$ models the population of Hungary, A, in millions, t years after 2006.

a. Find Hungary's population, in millions, for 2006, 2007, 2008, and 2009. Round to two decimal places.

b. Is Hungary's population increasing or decreasing?

149. The table shows the average amount that Americans paid for a gallon of gasoline from 2002 through 2006. Create a scatter plot for the data. Based on the shape of the scatter plot, would a logarithmic function, an exponential function, or a linear function be the best choice for modeling the data?

Average Gas Price in the U.S.

Year	Average Price per Gallon
2002	\$1.40
2003	\$1.60
2004	\$1.92
2005	\$2.30
2006	\$2.91

Source: Oil Price Information Service

150. **a.** Simplify: $e^{\ln 3}$.

b. Use your simplification from part (a) to rewrite 3^x in terms of base e.

Section 3.5 Exponential Growth and Decay; Modeling Data

Objectives

1. Model exponential growth and decay.
2. Use logistic growth models.
3. Use Newton's Law of Cooling.
4. Choose an appropriate model for data.
5. Express an exponential model in base *e*.

The most casual cruise on the Internet shows how people disagree when it comes to making predictions about the effects of the world's growing population. Some argue that there is a recent slowdown in the growth rate, economies remain robust, and famines in North Korea and Ethiopia are aberrations rather than signs of the future. Others say that the 6.8 billion people on Earth is twice as many as can be supported in middle-class comfort, and the world is running out of arable land and fresh water. Debates about entities that are growing exponentially can be approached mathematically: We can create functions that model data and use these functions to make predictions. In this section, we will show you how this is done.

1 Model exponential growth and decay.

Exponential Growth and Decay

One of algebra's many applications is to predict the behavior of variables. This can be done with *exponential growth* and *decay models*. With exponential growth or decay, quantities grow or decay at a rate directly proportional to their size.

Populations that are growing exponentially grow extremely rapidly as they get larger because there are more adults to have offspring. For example, world population is increasing at approximately 1.2% per year. This means that each year world population is 1.2% more than what it was in the previous year. In 2007, world population was 6.6 billion. Thus, we compute the world population in 2008 as follows:

$$6.6 \text{ billion} + 1.2\% \text{ of } 6.6 \text{ billion} = 6.6 + (0.012)(6.6) = 6.6792.$$

This computation indicates that 6.6792 billion people populated the world in 2008. The 0.0792 billion represents an increase of 79.2 million people from 2007 to 2008, the equivalent of the population of Germany. Using 1.2% as the annual rate of increase, world population for 2009 is found in a similar manner:

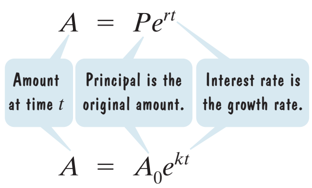
$$6.6792 + 1.2\% \text{ of } 6.6792 = 6.6792 + (0.012)(6.6792) \approx 6.759.$$

This computation indicates that approximately 6.759 billion people will populate the world in 2009.

The explosive growth of world population may remind you of the growth of money in an account subject to compound interest. The balance in an account subject to continuous compounding and world population are special cases of *exponential growth models*.

Study Tip

You have seen the formula for exponential growth before, but with different letters. It is the formula for compound interest with continous compounding.

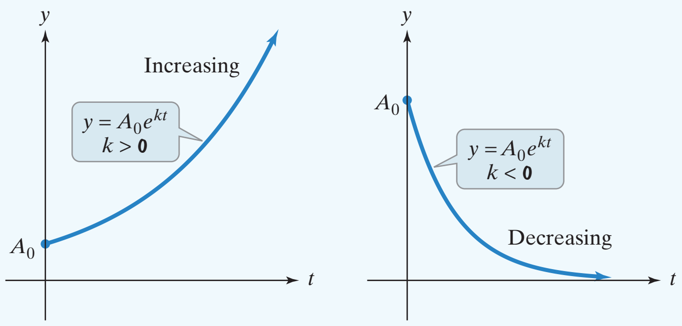


Exponential Growth and Decay Models

The mathematical model for **exponential growth** or **decay** is given by

$$f(t) = A_0e^{kt} \quad \text{or} \quad A = A_0e^{kt}.$$

- **If $k > 0$, the function models the amount, or size, of a *growing* entity.** A_0 is the original amount, or size, of the growing entity at time $t = 0$, A is the amount at time t, and k is a constant representing the growth rate.
- **If $k < 0$, the function models the amount, or size, of a *decaying* entity.** A_0 is the original amount, or size, of the decaying entity at time $t = 0$, A is the amount at time t, and k is a constant representing the decay rate.



(a) Exponential growth **(b)** Exponential decay

Sometimes we need to use given data to determine k, the rate of growth or decay. After we compute the value of k, we can use the formula $A = A_0e^{kt}$ to make predictions. This idea is illustrated in our first two examples.

EXAMPLE 1 Modeling the Growth of the U.S. Population

The graph in **Figure 3.22** shows the U.S. population, in millions, for five selected years from 1970 through 2007. In 1970, the U.S. population was 203.3 million. By 2007, it had grown to 300.9 million.

a. Find an exponential growth function that models the data for 1970 through 2007.

b. By which year will the U.S. population reach 315 million?

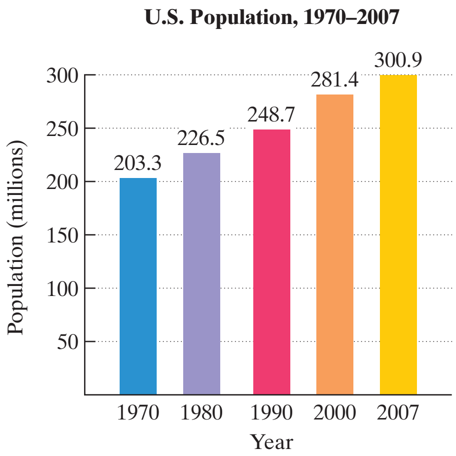


Figure 3.22 *Source:* U.S. Census Bureau

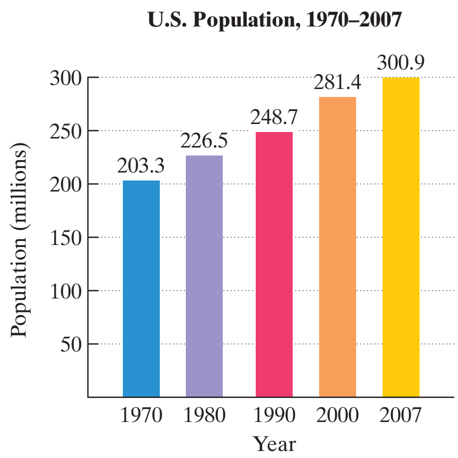


Figure 3.22 (repeated)

Solution

a. We use the exponential growth model

$$A = A_0e^{kt}$$

in which t is the number of years after 1970. This means that 1970 corresponds to $t = 0$. At that time the U.S. population was 203.3 million, so we substitute 203.3 for A_0 in the growth model:

$$A = 203.3e^{kt}.$$

We are given that 300.9 million was the population in 2007. Because 2007 is 37 years after 1970, when $t = 37$ the value of A is 300.9. Substituting these numbers into the growth model will enable us to find k, the growth rate. We know that $k > 0$ because the problem involves growth.

$$A = 203.3e^{kt}$$ Use the growth model with $A_0 = 203.3$.

$$300.9 = 203.3e^{k\cdot 37}$$ When $t = 37$, $A = 300.9$. Substitute these numbers into the model.

$$e^{37k} = \frac{300.9}{203.3}$$ Isolate the exponential factor by dividing both sides by 203.3. We also reversed the sides.

$$\ln e^{37k} = \ln\left(\frac{300.9}{203.3}\right)$$ Take the natural logarithm on both sides.

$$37k = \ln\left(\frac{300.9}{203.3}\right)$$ Simplify the left side using $\ln e^x = x$.

$$k = \frac{\ln\left(\frac{300.9}{203.3}\right)}{37} \approx 0.011$$ Divide both sides by 37 and solve for k. Then use a calculator.

The value of k, approximately 0.011, indicates a growth rate of about 1.1%. We substitute 0.011 for k in the growth model, $A = 203.3e^{kt}$, to obtain an exponential growth function for the U.S. population. It is

$$A = 203.3e^{0.011t},$$

where t is measured in years after 1970.

b. To find the year in which the U.S. population will reach 315 million, substitute 315 for A in the model from part (a) and solve for t.

$$A = 203.3e^{0.011t}$$ This is the model from part (a).

$$315 = 203.3e^{0.011t}$$ Substitute 315 for A.

$$e^{0.011t} = \frac{315}{203.3}$$ Divide both sides by 203.3. We also reversed the sides.

$$\ln e^{0.011t} = \ln\left(\frac{315}{203.3}\right)$$ Take the natural logarithm on both sides.

$$0.011t = \ln\left(\frac{315}{203.3}\right)$$ Simplify on the left using $\ln e^x = x$.

$$t = \frac{\ln\left(\frac{315}{203.3}\right)}{0.011} \approx 40$$ Divide both sides by 0.011 and solve for t. Then use a calculator.

Because t represents the number of years after 1970, the model indicates that the U.S. population will reach 315 million by 1970 + 40, or in the year 2010.

In Example 1, we used only two data values, the population for 1970 and the population for 2007, to develop a model for U.S. population growth from 1970 through 2007. By not using data for any other years, have we created a model that inaccurately describes both the existing data and future population projections given by the U.S. Census Bureau? Something else to think about: Is an exponential model the best choice for describing U.S. population growth, or might a linear model provide a better description? We return to these issues in Exercises 68–72 in the exercise set.

Check Point 1 In 1990, the population of Africa was 643 million and by 2006 it had grown to 906 million.

a. Use the exponential growth model $A = A_0e^{kt}$, in which t is the number of years after 1990, to find the exponential growth function that models the data.

b. By which year will Africa's population reach 2000 million, or two billion?

Our next example involves exponential decay and its use in determining the age of fossils and artifacts. The method is based on considering the percentage of carbon-14 remaining in the fossil or artifact. Carbon-14 decays exponentially with a *half-life* of approximately 5715 years. The **half-life** of a substance is the time required for half of a given sample to disintegrate. Thus, after 5715 years a given amount of carbon-14 will have decayed to half the original amount. Carbon dating is useful for artifacts or fossils up to 80,000 years old. Older objects do not have enough carbon-14 left to determine age accurately.

Carbon Dating and Artistic Development

The artistic community was electrified by the discovery in 1995 of spectacular cave paintings in a limestone cavern in France. Carbon dating of the charcoal from the site showed that the images, created by artists of remarkable talent, were 30,000 years old, making them the oldest cave paintings ever found. The artists seemed to have used the cavern's natural contours to heighten a sense of perspective. The quality of the painting suggests that the art of early humans did not mature steadily from primitive to sophisticated in any simple linear fashion.

EXAMPLE 2 Carbon-14 Dating: The Dead Sea Scrolls

a. Use the fact that after 5715 years a given amount of carbon-14 will have decayed to half the original amount to find the exponential decay model for carbon-14.

b. In 1947, earthenware jars containing what are known as the Dead Sea Scrolls were found by an Arab Bedouin herdsman. Analysis indicated that the scroll wrappings contained 76% of their original carbon-14. Estimate the age of the Dead Sea Scrolls.

Solution

a. We begin with the exponential decay model $A = A_0e^{kt}$. We know that $k < 0$ because the problem involves the decay of carbon-14. After 5715 years ($t = 5715$), the amount of carbon-14 present, A, is half the original amount, A_0. Thus, we can substitute $\frac{A_0}{2}$ for A in the exponential decay model. This will enable us to find k, the decay rate.

$$A = A_0e^{kt}$$ Begin with the exponential decay model.

$$\frac{A_0}{2} = A_0e^{k5715}$$ After 5715 years ($t = 5715$), $A = \frac{A_0}{2}$ (because the amount present, A, is half the original amount, A_0).

$$\frac{1}{2} = e^{5715k}$$ Divide both sides of the equation by A_0.

$$\ln\left(\frac{1}{2}\right) = \ln e^{5715k}$$ Take the natural logarithm on both sides.

$$\ln\left(\frac{1}{2}\right) = 5715k$$ Simplify the right side using $\ln e^x = x$.

$$k = \frac{\ln\left(\frac{1}{2}\right)}{5715} \approx -0.000121$$ Divide both sides by 5715 and solve for k.

Substituting -0.000121 for k in the decay model, $A = A_0e^{kt}$, the model for carbon-14 is

$$A = A_0e^{-0.000121t}.$$

b. In 1947, the Dead Sea Scrolls contained 76% of their original carbon-14. To find their age in 1947, substitute $0.76A_0$ for A in the model from part (a) and solve for t.

$$A = A_0e^{-0.000121t}$$ This is the decay model for carbon-14.

$$0.76A_0 = A_0e^{-0.000121t}$$ A, the amount present, is 76% of the original amount, so $A = 0.76A_0$.

$$0.76 = e^{-0.000121t}$$ Divide both sides of the equation by A_0.

$$\ln 0.76 = \ln e^{-0.000121t}$$ Take the natural logarithm on both sides.

$$\ln 0.76 = -0.000121t$$ Simplify the right side using $\ln e^x = x$.

$$t = \frac{\ln 0.76}{-0.000121} \approx 2268$$ Divide both sides by -0.000121 and solve for t.

The Dead Sea Scrolls are approximately 2268 years old plus the number of years between 1947 and the current year.

Check Point 2 Strontium-90 is a waste product from nuclear reactors. As a consequence of fallout from atmospheric nuclear tests, we all have a measurable amount of strontium-90 in our bones.

a. The half-life of strontium-90 is 28 years, meaning that after 28 years a given amount of the substance will have decayed to half the original amount. Find the exponential decay model for strontium-90.

b. Suppose that a nuclear accident occurs and releases 60 grams of strontium-90 into the atmosphere. How long will it take for strontium-90 to decay to a level of 10 grams?

2 Use logistic growth models.

Logistic Growth Models

From population growth to the spread of an epidemic, nothing on Earth can grow exponentially indefinitely. Growth is always limited. This is shown in **Figure 3.23** by the horizontal asymptote. The *logistic growth model* is a function used to model situations of this type.

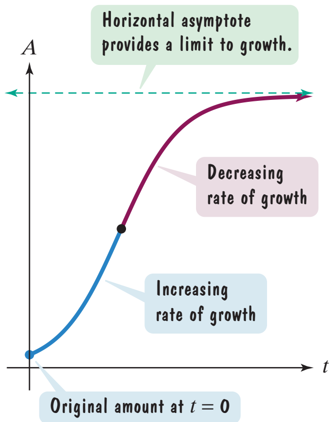


Figure 3.23 The logistic growth curve has a horizontal asymptote that identifies the limit of the growth of A over time.

Logistic Growth Model

The mathematical model for limited logistic growth is given by

$$f(t) = \frac{c}{1 + ae^{-bt}} \quad \text{or} \quad A = \frac{c}{1 + ae^{-bt}},$$

where a, b, and c are constants, with $c > 0$ and $b > 0$.

As time increases ($t \to \infty$), the expression ae^{-bt} in the model approaches 0, and A gets closer and closer to c. This means that $y = c$ is a horizontal asymptote for the graph of the function. Thus, the value of A can never exceed c and c represents the limiting size that A can attain.

EXAMPLE 3 Modeling the Spread of the Flu

The function

$$f(t) = \frac{30{,}000}{1 + 20e^{-1.5t}}$$

describes the number of people, $f(t)$, who have become ill with influenza t weeks after its initial outbreak in a town with 30,000 inhabitants.

a. How many people became ill with the flu when the epidemic began?

b. How many people were ill by the end of the fourth week?

c. What is the limiting size of $f(t)$, the population that becomes ill?

Solution

a. The time at the beginning of the flu epidemic is $t = 0$. Thus, we can find the number of people who were ill at the beginning of the epidemic by substituting 0 for t.

$$f(t) = \frac{30{,}000}{1 + 20e^{-1.5t}}$$ This is the given logistic growth function.

$$f(0) = \frac{30{,}000}{1 + 20e^{-1.5(0)}}$$ When the epidemic began, $t = 0$.

$$= \frac{30{,}000}{1 + 20}$$ $e^{-1.5(0)} = e^0 = 1$

$$\approx 1429$$

Approximately 1429 people were ill when the epidemic began.

b. We find the number of people who were ill at the end of the fourth week by substituting 4 for t in the logistic growth function.

$$f(t) = \frac{30{,}000}{1 + 20e^{-1.5t}}$$ Use the given logistic growth function.

$$f(4) = \frac{30{,}000}{1 + 20e^{-1.5(4)}}$$ To find the number of people ill by the end of week four, let $t = 4$.

$$\approx 28{,}583$$ Use a calculator.

Approximately 28,583 people were ill by the end of the fourth week. Compared with the number of people who were ill initially, 1429, this illustrates the virulence of the epidemic.

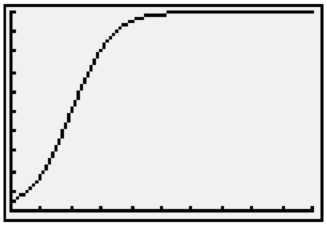
c. Recall that in the logistic growth model, $f(t) = \frac{c}{1 + ae^{-bt}}$, the constant c represents the limiting size that $f(t)$ can attain. Thus, the number in the numerator, 30,000, is the limiting size of the population that becomes ill.

Technology

The graph of the logistic growth function for the flu epidemic

$$y = \frac{30{,}000}{1 + 20e^{-1.5x}}$$

can be obtained using a graphing utility. We started x at 0 and ended at 10. This takes us to week 10. (In Example 3, we found that by week 4 approximately 28,583 people were ill.) We also know that 30,000 is the limiting size, so we took values of y up to 30,000. Using a [0, 10, 1] by [0, 30,000, 3000] viewing rectangle, the graph of the logistic growth function is shown below.



Check Point 3 In a learning theory project, psychologists discovered that

$$f(t) = \frac{0.8}{1 + e^{-0.2t}}$$

is a model for describing the proportion of correct responses, $f(t)$, after t learning trials.

a. Find the proportion of correct responses prior to learning trials taking place.

b. Find the proportion of correct responses after 10 learning trials.

c. What is the limiting size of $f(t)$, the proportion of correct responses, as continued learning trials take place?

3 Use Newton's Law of Cooling.

Modeling Cooling

Over a period of time, a cup of hot coffee cools to the temperature of the surrounding air. **Newton's Law of Cooling**, named after Sir Isaac Newton, states that the temperature of a heated object decreases exponentially over time toward the temperature of the surrounding medium.

Study Tip

Newton's Law of Cooling applies to any situation in which an object's temperature is different from that of the surrounding medium. Thus, it can be used to model the temperature of a heated object cooling to room temperature as well as the temperature of a frozen object thawing to room temperature.

Newton's Law of Cooling

The temperature, T, of a heated object at time t is given by

$$T = C + (T_0 - C)e^{kt},$$

where C is the constant temperature of the surrounding medium, T_0 is the initial temperature of the heated object, and k is a negative constant that is associated with the cooling object.

EXAMPLE 4 Using Newton's Law of Cooling

A cake removed from the oven has a temperature of 210°F. It is left to cool in a room that has a temperature of 70°F. After 30 minutes, the temperature of the cake is 140°F.

a. Use Newton's Law of Cooling to find a model for the temperature of the cake, T, after t minutes.

b. What is the temperature of the cake after 40 minutes?

c. When will the temperature of the cake be 90°F?

Solution

a. We use Newton's Law of Cooling

$$T = C + (T_0 - C)e^{kt}.$$

When the cake is removed from the oven, its temperature is 210°F. This is its initial temperature: $T_0 = 210$. The constant temperature of the room is 70°F: $C = 70$. Substitute these values into Newton's Law of Cooling. Thus, the temperature of the cake, T, in degrees Fahrenheit, at time t, in minutes, is

$$T = 70 + (210 - 70)e^{kt} = 70 + 140e^{kt}.$$

After 30 minutes, the temperature of the cake is 140°F. This means that when $t = 30$, $T = 140$. Substituting these numbers into Newton's Law of Cooling will enable us to find k, a negative constant.

$$T = 70 + 140e^{kt}$$ Use Newton's Law of Cooling from above.

$$140 = 70 + 140e^{k \cdot 30}$$ When $t = 30$, $T = 140$. Substitute these numbers into the cooling model.

$$70 = 140e^{30k}$$ Subtract 70 from both sides.

$$e^{30k} = \frac{1}{2}$$ Isolate the exponential factor by dividing both sides by 140. We also reversed the sides.

$$\ln e^{30k} = \ln\left(\frac{1}{2}\right)$$ Take the natural logarithm on both sides.

$$30k = \ln\left(\frac{1}{2}\right)$$ Simplify the left side using $\ln e^x = x$.

$$k = \frac{\ln\left(\frac{1}{2}\right)}{30} \approx -0.0231$$ Divide both sides by 30 and solve for k.

We substitute -0.0231 for k into Newton's Law of Cooling, $T = 70 + 140e^{kt}$. The temperature of the cake, T, in degrees Fahrenheit, after t minutes is modeled by

$$T = 70 + 140e^{-0.0231t}.$$

b. To find the temperature of the cake after 40 minutes, we substitute 40 for t into the cooling model from part (a) and evaluate to find T.

$$T = 70 + 140e^{-0.0231(40)} \approx 126$$

After 40 minutes, the temperature of the cake will be approximately 126°F.

c. To find when the temperature of the cake will be 90°F, we substitute 90 for T into the cooling model from part (a) and solve for t.

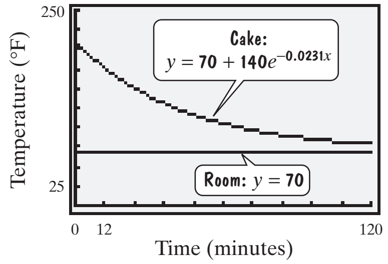
$T = 70 + 140e^{-0.0231t}$	This is the cooling model from part (a).
$90 = 70 + 140e^{-0.0231t}$	Substitute 90 for T.
$20 = 140e^{-0.0231t}$	Subtract 70 from both sides.
$e^{-0.0231t} = \frac{1}{7}$	Divide both sides by 140. We also reversed the sides.
$\ln e^{-0.0231t} = \ln\left(\frac{1}{7}\right)$	Take the natural logarithm on both sides.
$-0.0231t = \ln\left(\frac{1}{7}\right)$	Simplify the left side using $\ln e^x = x$.
$t = \dfrac{\ln\left(\frac{1}{7}\right)}{-0.0231} \approx 84$	Solve for t by dividing both sides by -0.0231.

The temperature of the cake will be 90°F after approximately 84 minutes.

Technology

Graphic Connections

The graphs illustrate how the temperature of the cake decreases exponentially over time toward the 70°F room temperature.



Check Point 4 An object is heated to 100°C. It is left to cool in a room that has a temperature of 30°C. After 5 minutes, the temperature of the object is 80°C.

a. Use Newton's Law of Cooling to find a model for the temperature of the object, T, after t minutes.

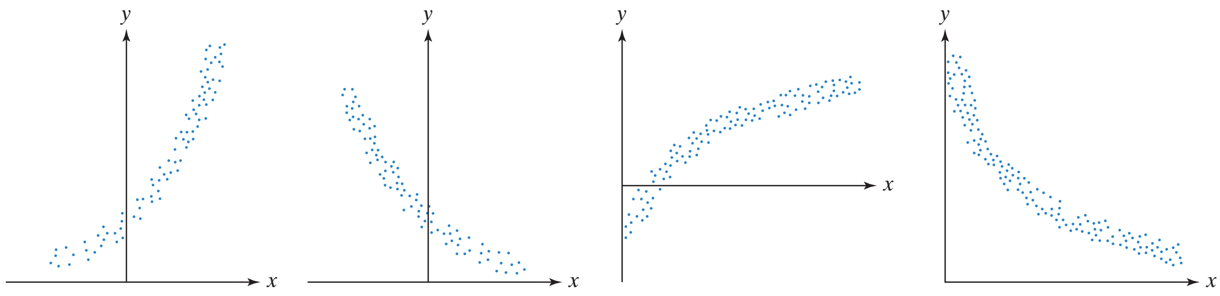
b. What is the temperature of the object after 20 minutes?

c. When will the temperature of the object be 35°C?

4 Choose an appropriate model for data.

Modeling Data

Throughout this chapter, we have been working with models that were given. However, we can create functions that model data by observing patterns in scatter plots. **Figure 3.24** shows scatter plots for data that are exponential or logarithmic.



$y = ab^x, a > 0, b > 1$ Exponential | $y = ab^x, a > 0, 0 < b < 1$ Exponential | $y = a + b \ln x, a > 0, b > 0$ Logarithmic | $y = a + b \ln x, a > 0, b < 0$ Logarithmic

Figure 3.24 Scatter plots for exponential or logarithmic models

EXAMPLE 5 Choosing a Model for Data

The bar graph in **Figure 3.25(a)** indicates that for the period from 2002 through 2006, an increasing number of Americans had weight-loss surgery. A scatter plot is shown in **Figure 3.25(b)**. What type of function would be a good choice for modeling the data?

Number of Weight-Loss Surgeries in the United States

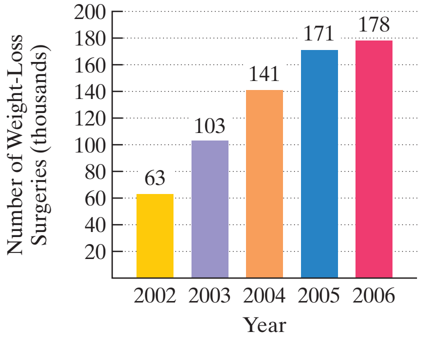


Figure 3.25(a)
Source: American Society for Bariatric Surgery

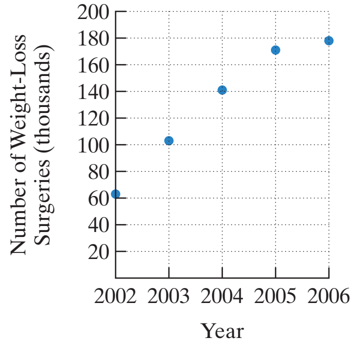


Figure 3.25(b)

Solution Because the data in the scatter plot increase rapidly at first and then begin to level off a bit, the shape suggests that a logarithmic function is a good choice for modeling the data.

Table 3.5 Population and Walking Speed

Population (thousands)	Walking Speed (feet per second)
5.5	0.6
14	1.0
71	1.6
138	1.9
342	2.2

Source: Mark and Helen Bornstein, "The Pace of Life"

Check Point 5 **Table 3.5** shows the populations of various cities, in thousands, and the average walking speed, in feet per second, of a person living in the city. Create a scatter plot for the data. Based on the scatter plot, what function would be a good choice for modeling the data?

How can we obtain a logarithmic function that models the data for the number of weight-loss surgeries, in thousands, shown in **Figure 3.25(a)**? A graphing utility can be used to obtain a logarithmic model of the form $y = a + b \ln x$. **Because the domain of the logarithmic function is the set of positive numbers, zero must not be a value for *x*.** What does this mean for the number of weight-loss surgeries that begin in the year 2002? We must start values of x after 0. Thus, we'll assign x to represent the number of years after 2001. This gives us the data shown in **Table 3.6**. Using the logarithmic regression option, we obtain the equation in **Figure 3.26**.

Table 3.6

x, Number of Years after 2001	*y*, Number of Weight-Loss Surgeries (thousands)
1 (2002)	63
2 (2003)	103
3 (2004)	141
4 (2005)	171
5 (2006)	178

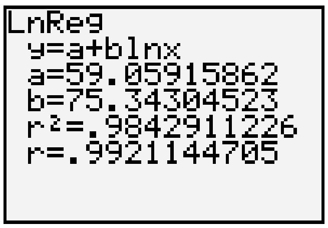


Figure 3.26 A logarithmic model for the data in **Table 3.6**

From **Figure 3.26**, we see that the logarithmic model of the data, with numbers rounded to three decimal places, is

$$y = 59.059 + 75.343 \ln x.$$

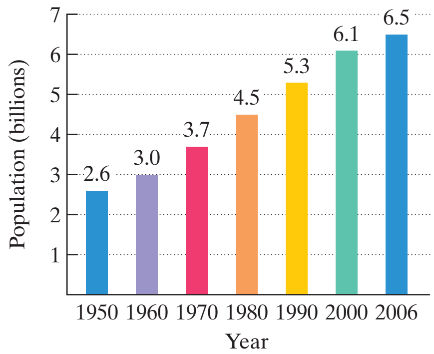
The number r that appears in **Figure 3.26** is called the **correlation coefficient** and is a measure of how well the model fits the data. The value of r is such that

$-1 \le r \le 1$. A positive r means that as the x-values increase, so do the y-values. A negative r means that as the x-values increase, the y-values decrease. **The closer that r is to -1 or 1, the better the model fits the data.** Because r is approximately 0.992, the model fits the data very well.

EXAMPLE 6 Choosing a Model for Data

Figure 3.27(a) shows world population, in billions, for seven selected years from 1950 through 2006. A scatter plot is shown in **Figure 3.27(b)**. Suggest two types of functions that would be good choices for modeling the data.

World Population, 1950–2006



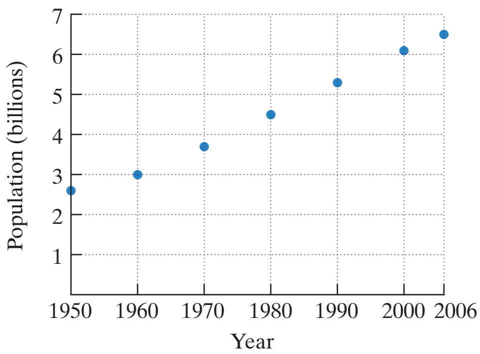


Figure 3.27(a)
Source: U.S. Census Bureau, International Database

Figure 3.27(b)

Solution Because the data in the scatter plot appear to increase more and more rapidly, the shape suggests that an exponential model might be a good choice. Furthermore, we can probably draw a line that passes through or near the seven points. Thus, a linear function would also be a good choice for modeling the data.

The data for world population are shown in **Table 3.7**. Using a graphing utility's linear regression feature and exponential regression feature, we enter the data and obtain the models shown in **Figure 3.28**.

Although the domain of $y = ab^x$ is the set of all real numbers, some graphing utilities only accept positive values for x. That's why we assigned x to represent the number of years after 1949.

Table 3.7

x, Number of Years after 1949	y, World Population (billions)
1 (1950)	2.6
11 (1960)	3.0
21 (1970)	3.7
31 (1980)	4.5
41 (1990)	5.3
51 (2000)	6.1
57 (2006)	6.5

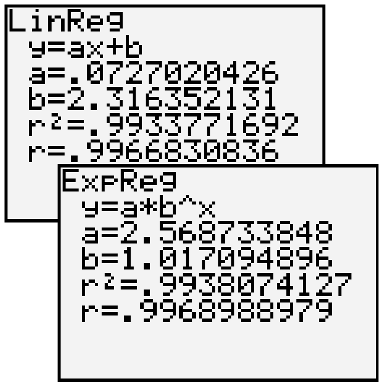


Figure 3.28 A linear model and an exponential model for the data in **Table 3.7**

Study Tip

Once you have obtained one or more models for data, you can use a graphing utility's TABLE feature to numerically see how well each model describes the data. Enter the models as y_1, y_2, and so on. Create a table, scroll through the table, and compare the table values given by the models to the actual data.

Using **Figure 3.28** and rounding to three decimal places, the functions

$$f(x) = 0.073x + 2.316 \quad \text{and} \quad g(x) = 2.569(1.017)^x$$

model world population, in billions, x years after 1949. We named the linear function f and the exponential function g, although any letters can be used. Because r, the correlation coefficient, is close to 1 in each screen in **Figure 3.28**, the models fit the data very well.

Check Point 6 It's a new dawn for technology, appealing to people who have just hung that huge plasma in the home theater: the era of high-definition television. **Table 3.8** shows the percentage of households in the United States with HDTV sets for four selected years from 2001 through 2009. Create a scatter plot for the data. Based on the scatter plot, what function would be a good choice for modeling the data?

Table 3.8 Percentage of U.S. Households with HDTV sets

Year	Percentage of Households
2001	1%
2004	10%
2006	26%
2009	47%

Source: Consumer Electronics Association

When using a graphing utility to model data, begin with a scatter plot, drawn either by hand or with the graphing utility, to obtain a general picture for the shape of the data. It might be difficult to determine which model best fits the data—linear, logarithmic, exponential, quadratic, or something else. If necessary, use your graphing utility to fit several models to the data. The best model is the one that yields the value of r, the correlation coefficient, closest to 1 or -1. Finding a proper fit for data can be almost as much art as it is mathematics. In this era of technology, the process of creating models that best fit data is one that involves more decision making than computation.

5 Express an exponential model in base e.

Expressing $y = ab^x$ in Base e

Graphing utilities display exponential models in the form $y = ab^x$. However, our discussion of exponential growth involved base e. Because of the inverse property $b = e^{\ln b}$, we can rewrite any model in the form $y = ab^x$ in terms of base e.

Expressing an Exponential Model in Base e

$$y = ab^x \quad \text{is equivalent to} \quad y = ae^{(\ln b)\cdot x}.$$

EXAMPLE 7 **Rewriting the Model for World Population in Base e**

We have seen that the function

$$g(x) = 2.569(1.017)^x$$

models world population, $g(x)$, in billions, x years after 1949. Rewrite the model in terms of base e.

Solution We use the two equivalent equations shown in the voice balloons to rewrite the model in terms of base e.

$y = ab^x$ $\qquad$ $y = ae^{(\ln b)\cdot x}$

$$g(x) = 2.569(1.017)^x \quad \text{is equivalent to} \quad g(x) = 2.569e^{(\ln 1.017)\cdot x}.$$

Using $\ln 1.017 \approx 0.017$, the exponential growth model for world population, $g(x)$, in billions, x years after 1949 is

$$g(x) = 2.569e^{0.017x}.$$

In Example 7, we can replace $g(x)$ with A and x with t so that the model has the same letters as those in the exponential growth model $A = A_0e^{kt}$.

$$A = A_o e^{kt}$$ This is the exponential growth model.

$$A = 2.569e^{0.017t}$$ This is the model for world population.

The value of k, 0.017, indicates a growth rate of 1.7% per year. Although this is an excellent model for the data, we must be careful about making projections about world population using this growth function. Why? World population growth rate is now 1.2%, not 1.7%, so our model will overestimate future populations.

Check Point 7 Rewrite $y = 4(7.8)^x$ in terms of base e. Express the answer in terms of a natural logarithm and then round to three decimal places.

Exercise Set 3.5

Practice Exercises and Application Exercises

The exponential models describe the population of the indicated country, A, in millions, t years after 2006. Use these models to solve Exercises 1–6.

India $A = 1095.4e^{0.014t}$

Iraq $A = 26.8e^{0.027t}$

Japan $A = 127.5e^{0.001t}$

Russia $A = 142.9e^{-0.004t}$

1. What was the population of Japan in 2006?

2. What was the population of Iraq in 2006?

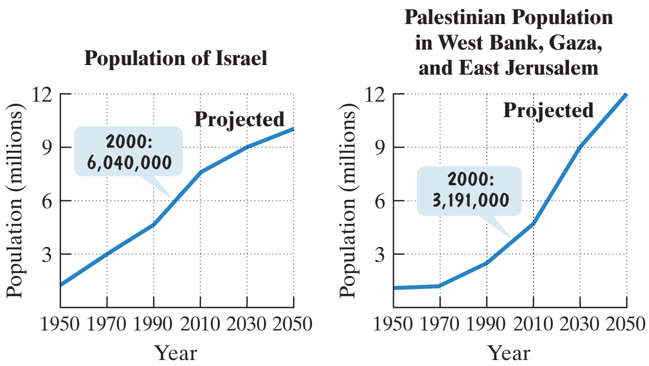
3. Which country has the greatest growth rate? By what percentage is the population of that country increasing each year?

4. Which country has a decreasing population? By what percentage is the population of that country decreasing each year?

5. When will India's population be 1238 million?

6. When will India's population be 1416 million?

About the size of New Jersey, Israel has seen its population soar to more than 6 million since it was established. With the help of U.S. aid, the country now has a diversified economy rivaling those of other developed Western nations. By contrast, the Palestinians, living under Israeli occupation and a corrupt regime, endure bleak conditions. The graphs show that by 2050, Palestinians in the West Bank, Gaza Strip, and East Jerusalem will outnumber Israelis. Exercises 7–8 involve the projected growth of these two populations.



Source: Newsweek

7. a. In 2000, the population of Israel was approximately 6.04 million and by 2050 it is projected to grow to 10 million. Use the exponential growth model $A = A_0e^{kt}$, in which t is the number of years after 2000, to find an exponential growth function that models the data.

b. In which year will Israel's population be 9 million?

8. a. In 2000, the population of the Palestinians in the West Bank, Gaza Strip, and East Jerusalem was approximately 3.2 million and by 2050 it is projected to grow to 12 million. Use the exponential growth model $A = A_0e^{kt}$, in which t is the number of years after 2000, to find the exponential growth function that models the data.

b. In which year will the Palestinian population be 9 million?

In Exercises 9–14, complete the table. Round projected populations to one decimal place and values of k to four decimal places.

	Country	2007 Population (millions)	Projected 2025 Population (millions)	Projected Growth Rate, *k*
9.	Philippines	91.1		0.0147
10.	Pakistan	164.7		0.0157
11.	Colombia	44.4	55.2	
12.	Madagascar	19.4	32.4	
13.	South Africa	44.0	40.0	
14.	Bulgaria	7.3	6.3	

Source: International Programs Center, U.S. Census Bureau

An artifact originally had 16 grams of carbon-14 present. The decay model $A = 16e^{-0.000121t}$ describes the amount of carbon-14 present after t years. Use this model to solve Exercises 15–16.

15. How many grams of carbon-14 will be present in 5715 years?

16. How many grams of carbon-14 will be present in 11,430 years?

17. The half-life of the radioactive element krypton-91 is 10 seconds. If 16 grams of krypton-91 are initially present, how many grams are present after 10 seconds? 20 seconds? 30 seconds? 40 seconds? 50 seconds?

18. The half-life of the radioactive element plutonium-239 is 25,000 years. If 16 grams of plutonium-239 are initially present, how many grams are present after 25,000 years? 50,000 years? 75,000 years? 100,000 years? 125,000 years?

Use the exponential decay model for carbon-14, $A = A_0e^{-0.000121t}$, to solve Exercises 19–20.

19. Prehistoric cave paintings were discovered in a cave in France. The paint contained 15% of the original carbon-14. Estimate the age of the paintings.

20. Skeletons were found at a construction site in San Francisco in 1989. The skeletons contained 88% of the expected amount of carbon-14 found in a living person. In 1989, how old were the skeletons?

In Exercises 21–26, complete the table. Round half-lives to one decimal place and values of k to six decimal places.

	Radioactive Substance	Half-Life	Decay Rate, *k*
21.	Tritium		5.5% per year $= -0.055$
22.	Krypton-85		6.3% per year $= -0.063$
23.	Radium-226	1620 years	
24.	Uranium-238	4560 years	
25.	Arsenic-74	17.5 days	
26.	Calcium-47	113 hours	

27. The August 1978 issue of *National Geographic* described the 1964 find of bones of a newly discovered dinosaur weighing 170 pounds, measuring 9 feet, with a 6-inch claw on one toe of each hind foot. The age of the dinosaur was estimated using potassium-40 dating of rocks surrounding the bones.

a. Potassium-40 decays exponentially with a half-life of approximately 1.31 billion years. Use the fact that after 1.31 billion years a given amount of potassium-40 will have decayed to half the original amount to show that the decay model for potassium-40 is given by $A = A_0e^{-0.52912t}$, where t is in billions of years.

b. Analysis of the rocks surrounding the dinosaur bones indicated that 94.5% of the original amount of potassium-40 was still present. Let $A = 0.945A_0$ in the model in part (a) and estimate the age of the bones of the dinosaur.

Use the exponential decay model, $A = A_0e^{kt}$, to solve Exercises 28–31. Round answers to one decimal place.

28. The half-life of thorium-229 is 7340 years. How long will it take for a sample of this substance to decay to 20% of its original amount?

29. The half-life of lead is 22 years. How long will it take for a sample of this substance to decay to 80% of its original amount?

30. The half-life of aspirin in your bloodstream is 12 hours. How long will it take for the aspirin to decay to 70% of the original dosage?

31. Xanax is a tranquilizer used in the short-term relief of symptoms of anxiety. Its half-life in the bloodstream is 36 hours. How long will it take for Xanax to decay to 90% of the original dosage?

32. A bird species in danger of extinction has a population that is decreasing exponentially ($A = A_0e^{kt}$). Five years ago the population was at 1400 and today only 1000 of the birds are alive. Once the population drops below 100, the situation will be irreversible. When will this happen?

33. Use the exponential growth model, $A = A_0e^{kt}$, to show that the time it takes a population to double (to grow from A_0 to $2A_0$) is given by $t = \frac{\ln 2}{k}$.

34. Use the exponential growth model, $A = A_0e^{kt}$, to show that the time it takes a population to triple (to grow from A_0 to $3A_0$) is given by $t = \frac{\ln 3}{k}$.

Use the formula $t = \frac{\ln 2}{k}$ that gives the time for a population with a growth rate k to double to solve Exercises 35–36. Express each answer to the nearest whole year.

35. The growth model $A = 4.1e^{0.01t}$ describes New Zealand's population, A, in millions, t years after 2006.

a. What is New Zealand's growth rate?

b. How long will it take New Zealand to double its population?

36. The growth model $A = 107.4e^{0.012t}$ describes Mexico's population, A, in millions, t years after 2006.

a. What is Mexico's growth rate?

b. How long will it take Mexico to double its population?

37. The logistic growth function

$$f(t) = \frac{100{,}000}{1 + 5000e^{-t}}$$

describes the number of people, $f(t)$, who have become ill with influenza t weeks after its initial outbreak in a particular community.

a. How many people became ill with the flu when the epidemic began?

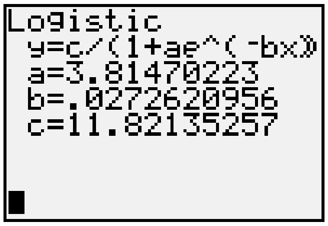
b. How many people were ill by the end of the fourth week?

c. What is the limiting size of the population that becomes ill?

Shown, again, is world population, in billions, for seven selected years from 1950 through 2006.

x, Number of Years after 1949	*y*, World Population (billions)
1 (1950)	2.6
11 (1960)	3.0
21 (1970)	3.7
31 (1980)	4.5
41 (1990)	5.3
51 (2000)	6.1
57 (2006)	6.5

Using a graphing utility's logistic regression option, we obtain the equation shown on the screen.



We see from the calculator screen that a logistic growth model for world population, $f(x)$, in billions, x years after 1949 is

$$f(x) = \frac{11.82}{1 + 3.81e^{-0.027x}}.$$

Use this function to solve Exercises 38–42.

38. How well does the function model the data for 2000?

39. How well does the function model the data for 2006?

40. When will world population reach 7 billion?

41. When will world population reach 8 billion?

42. According to the model, what is the limiting size of the population that Earth will eventually sustain?

The logistic growth function

$$P(x) = \frac{90}{1 + 271e^{-0.122x}}$$

models the percentage, $P(x)$, of Americans who are x years old with some coronary heart disease. Use the function to solve Exercises 43–46.

43. What percentage of 20-year-olds have some coronary heart disease?

44. What percentage of 80-year-olds have some coronary heart disease?

45. At what age is the percentage of some coronary heart disease 50%?

46. At what age is the percentage of some coronary heart disease 70%?

Use Newton's Law of Cooling, $T = C + (T_0 - C)e^{kt}$, to solve Exercises 47–50.

47. A bottle of juice initially has a temperature of 70°F. It is left to cool in a refrigerator that has a temperature of 45°F. After 10 minutes, the temperature of the juice is 55°F.

a. Use Newton's Law of Cooling to find a model for the temperature of the juice, T, after t minutes.

b. What is the temperature of the juice after 15 minutes?

c. When will the temperature of the juice be 50°F?

48. A pizza removed from the oven has a temperature of 450°F. It is left sitting in a room that has a temperature of 70°F. After 5 minutes, the temperature of the pizza is 300°F.

a. Use Newton's Law of Cooling to find a model for the temperature of the pizza, T, after t minutes.

b. What is the temperature of the pizza after 20 minutes?

c. When will the temperature of the pizza be 140°F?

49. A frozen steak initially has a temperature of 28°F. It is left to thaw in a room that has a temperature of 75°F. After 10 minutes, the temperature of the steak has risen to 38°F. After how many minutes will the temperature of the steak be 50°F?

50. A frozen steak initially has a temperature of 24°F. It is left to thaw in a room that has a temperature of 65°F. After 10 minutes, the temperature of the steak has risen to 30°F. After how many minutes will the temperature of the steak be 45°F?

Exercises 51–54 present data in the form of tables. For each data set shown by the table,

a. *Create a scatter plot for the data.*

b. *Use the scatter plot to determine whether an exponential function or a logarithmic function is the best choice for modeling the data. (If applicable, in Exercise 74, you will use your graphing utility to obtain these functions.)*

51. Percent of Miscarriages, by Age

Woman's Age	Percent of Miscarriages
22	9%
27	10%
32	13%
37	20%
42	38%
47	52%

Source: *Time*

52. Savings Needed for Health-Care Expenses during Retirement

Age at Death	Savings Needed
80	\$219,000
85	\$307,000
90	\$409,000
95	\$524,000
100	\$656,000

Source: Employee Benefit Research Institute

53. Intensity and Loudness Level of Various Sounds

Intensity (watts per meter^2)	Loudness Level (decibels)
0.1 (loud thunder)	110
1 (rock concert, 2 yd from speakers)	120
10 (jackhammer)	130
100 (jet takeoff, 40 yd away)	140

54. Temperature Increase in an Enclosed Vehicle

Minutes	Temperature Increase (°F)
10	19°
20	29°
30	34°
40	38°
50	41°
60	43°

In Exercises 55–58, rewrite the equation in terms of base e. Express the answer in terms of a natural logarithm and then round to three decimal places.

55. $y = 100(4.6)^x$

56. $y = 1000(7.3)^x$

57. $y = 2.5(0.7)^x$

58. $y = 4.5(0.6)^x$

Writing in Mathematics

59. Nigeria has a growth rate of 0.025 or 2.5%. Describe what this means.

60. How can you tell whether an exponential model describes exponential growth or exponential decay?

61. Suppose that a population that is growing exponentially increases from 800,000 people in 2003 to 1,000,000 people in 2006. Without showing the details, describe how to obtain the exponential growth function that models the data.

62. What is the half-life of a substance?

63. Describe a difference between exponential growth and logistic growth.

64. Describe the shape of a scatter plot that suggests modeling the data with an exponential function.

65. You take up weightlifting and record the maximum number of pounds you can lift at the end of each week. You start off with rapid growth in terms of the weight you can lift from week to week, but then the growth begins to level off. Describe how to obtain a function that models the number of pounds you can lift at the end of each week. How can you use this function to predict what might happen if you continue the sport?

66. Would you prefer that your salary be modeled exponentially or logarithmically? Explain your answer.

67. One problem with all exponential growth models is that nothing can grow exponentially forever. Describe factors that might limit the size of a population.

Technology Exercises

In Example 1 on page 437, we used two data points and an exponential function to model the population of the United States from 1970 through 2007. The data are shown again in the table. Use all five data points to solve Exercises 68–72.

x, Number of Years after 1969	y, U.S. Population (millions)
1 (1970)	203.3
11 (1980)	226.5
21 (1990)	248.7
31 (2000)	281.4
38 (2007)	300.9

68. a. Use your graphing utility's exponential regression option to obtain a model of the form $y = ab^x$ that fits the data. How well does the correlation coefficient, r, indicate that the model fits the data?

b. Rewrite the model in terms of base e. By what percentage is the population of the United States increasing each year?

69. Use your graphing utility's logarithmic regression option to obtain a model of the form $y = a + b \ln x$ that fits the data. How well does the correlation coefficient, r, indicate that the model fits the data?

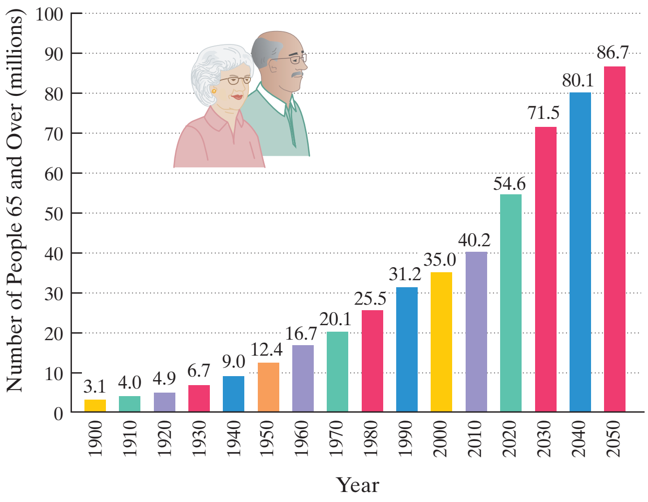
70. Use your graphing utility's linear regression option to obtain a model of the form $y = ax + b$ that fits the data. How well does the correlation coefficient, r, indicate that the model fits the data?

71. Use your graphing utility's power regression option to obtain a model of the form $y = ax^b$ that fits the data. How well does the correlation coefficient, r, indicate that the model fits the data?

72. Use the values of r in Exercises 68–71 to select the two models of best fit. Use each of these models to predict by which year the U.S. population will reach 315 million. How do these answers compare to the year we found in Example 1, namely 2010? If you obtained different years, how do you account for this difference?

73. The figure shows the number of people in the United States age 65 and over, with projected figures for the year 2010 and beyond.

United States Population Age 65 and Over



Source: U.S. Census Bureau

a. Let x represent the number of years after 1899 and let y represent the U.S. population age 65 and over, in millions. Use your graphing utility to find the model that best fits the data in the bar graph.

b. Rewrite the model in terms of base e. By what percentage is the 65 and over population increasing each year?

74. In Exercises 51–54, you determined the best choice for the kind of function that modeled the data in the table. For each of the exercises that you worked, use a graphing utility to find the actual function that best fits the data. Then use the model to make a reasonable prediction for a value that exceeds those shown in the table's first column.

Critical Thinking Exercises

Make Sense? *In Exercises 75–78, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

75. I used an exponential model with a positive growth rate to describe the depreciation in my car's value over four years.

76. After 100 years, a population whose growth rate is 3% will have three times as many people as a population whose growth rate is 1%.

77. When I used an exponential function to model Russia's declining population, the growth rate k was negative.

78. Because carbon-14 decays exponentially, carbon dating can determine the ages of ancient fossils.

The exponential growth models describe the population of the indicated country, A, in millions, t years after 2006.

Canada $A = 33.1e^{0.009t}$

Uganda $A = 28.2e^{0.034t}$

In Exercises 79–82, use this information to determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

79. In 2006, Canada's population exceeded Uganda's by 4.9 million.

80. By 2009, the models indicate that Canada's population will exceed Uganda's by approximately 2.8 million.

81. The models indicate that in 2013, Uganda's population will exceed Canada's.

82. Uganda's growth rate is approximately 3.8 times that of Canada's.

83. Use Newton's Law of Cooling, $T = C + (T_0 - C)e^{kt}$, to solve this exercise. At 9:00 A.M., a coroner arrived at the home of a person who had died during the night. The temperature of the room was 70°F, and at the time of death the person had a body temperature of 98.6°F. The coroner took the body's temperature at 9:30 A.M., at which time it was 85.6°F, and again at 10:00 A.M., when it was 82.7°F. At what time did the person die?

Group Exercise

84. Each group member should consult an almanac, newspaper, magazine, or the Internet to find data that can be modeled by exponential or logarithmic functions. Group members should select the two sets of data that are most interesting and relevant. For each set selected, find a model that best fits the data. Each group member should make one prediction based on the model and then discuss a consequence of this prediction. What factors might change the accuracy of the prediction?

Preview Exercises

Exercises 85–87 will help you prepare for the material covered in the first section of the next chapter.

85. Solve: $\frac{5\pi}{4} = 2\pi x$.

86. Simplify: $\frac{17\pi}{6} - 2\pi$.

87. Simplify: $-\frac{\pi}{12} + 2\pi$.