Chapter 3 Mid-Chapter Check Point

What You Know: We evaluated and graphed exponential functions $[f(x) = b^x, b > 0 \text{ and } b \neq 1]$, including the natural exponential function $[f(x) = e^x, e \approx 2.718]$. A function has an inverse that is a function if there is no horizontal line that intersects the function's graph more than once. The exponential function passes this horizontal line test and we called the inverse of the exponential function with base b the logarithmic function with base b. We learned that $y = \log_b x$ is equivalent to $b^y = x$. We evaluated and graphed logarithmic functions, including the common logarithmic function $[f(x) = \log_{10} x$ or $f(x) = \log x]$ and the natural logarithmic function $[f(x) = \log_e x$ or $f(x) = \ln x]$. We learned to use transformations to graph exponential and logarithmic functions. Finally, we used properties of logarithms to expand and condense logarithmic expressions.

In Exercises 1–5, graph f and g in the same rectangular coordinate system. Graph and give equations of all asymptotes. Give each function's domain and range.

1. $f(x) = 2^x$ and $g(x) = 2^x - 3$
2. $f(x) = \left(\frac{1}{2}\right)^x$ and $g(x) = \left(\frac{1}{2}\right)^{x-1}$
3. $f(x) = e^x$ and $g(x) = \ln x$
4. $f(x) = \log_2 x$ and $g(x) = \log_2(x - 1) + 1$
5. $g(x) = \log_{\frac{1}{2}} x$ and $g(x) = -2 \log_{\frac{1}{2}} x$

In Exercises 6–9, find the domain of each function.

6. $f(x) = \log_3(x + 6)$
7. $g(x) = \log_3 x + 6$
8. $h(x) = \log_3(x + 6)^2$
9. $f(x) = 3^{x+6}$

In Exercises 10–20, evaluate each expression without using a calculator. If evaluation is not possible, state the reason.

10. $\log_2 8 + \log_5 25$
11. $\log_3 \frac{1}{9}$
12. $\log_{100} 10$
13. $\log \sqrt[3]{10}$
14. $\log_2(\log_3 81)$
15. $\log_3\left(\log_2 \frac{1}{8}\right)$
16. $6^{\log_6 5}$
17. $\ln e^{\sqrt{7}}$
18. $10^{\log 13}$
19. $\log_{100} 0.1$
20. $\log_\pi \pi^{\sqrt{\pi}}$

In Exercises 21–22, expand and evaluate numerical terms.

21. $\log\left(\frac{\sqrt{xy}}{1000}\right)$
22. $\ln(e^{19}x^{20})$

In Exercises 23–25, write each expression as a single logarithm.

23. $8 \log_7 x - \frac{1}{3}\log_7 y$
24. $7 \log_5 x + 2 \log_5 x$
25. $\frac{1}{2}\ln x - 3 \ln y - \ln(z - 2)$
26. Use the formulas

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \quad \text{and} \quad A = Pe^{rt}$$

to solve this exercise. You decide to invest \$8000 for 3 years at an annual rate of 8%. How much more is the return if the interest is compounded continuously than if it is compounded monthly? Round to the nearest dollar.

Section 3.4 Exponential and Logarithmic Equations

Objectives

1. Use like bases to solve exponential equations.
2. Use logarithms to solve exponential equations.
3. Use the definition of a logarithm to solve logarithmic equations.
4. Use the one-to-one property of logarithms to solve logarithmic equations.
5. Solve applied problems involving exponential and logarithmic equations.

At age 20, you inherit \$30,000. You'd like to put aside \$25,000 and eventually have over half a million dollars for early retirement. Is this possible? In this section, you will see how techniques for solving equations with variable exponents provide an answer to this question.

Exponential Equations

An **exponential equation** is an equation containing a variable in an exponent. Examples of exponential equations include

$$2^{3x-8} = 16, \qquad 4^x = 15, \quad \text{and} \quad 40e^{0.6x} = 240.$$

1 Use like bases to solve exponential equations.

Some exponential equations can be solved by expressing each side of the equation as a power of the same base. All exponential functions are one-to-one—that is, no two different ordered pairs have the same second component. Thus, if b is a positive number other than 1 and $b^M = b^N$, then $M = N$.

Solving Exponential Equations by Expressing Each Side as a Power of the Same Base

$$\text{If } b^M = b^N, \text{ then } M = N.$$

Express each side as a power of the same base.

Set the exponents equal to each other.

1. Rewrite the equation in the form $b^M = b^N$.
2. Set $M = N$.
3. Solve for the variable.

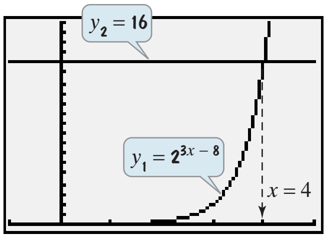
Technology

Graphic Connections

The graphs of

$$y_1 = 2^{3x-8}$$
$$\text{and} \quad y_2 = 16$$

have an intersection point whose x-coordinate is 4. This verifies that $\{4\}$ is the solution set of $2^{3x-8} = 16$.



$[-1, 5, 1]$ by $[0, 20, 1]$

EXAMPLE 1 Solving Exponential Equations

Solve: **a.** $2^{3x-8} = 16$ **b.** $27^{x+3} = 9^{x-1}$.

Solution In each equation, express both sides as a power of the same base. Then set the exponents equal to each other and solve for the variable.

a. Because 16 is 2^4, we express each side of $2^{3x-8} = 16$ in terms of base 2.

$2^{3x-8} = 16$ This is the given equation.

$2^{3x-8} = 2^4$ Write each side as a power of the same base.

$3x - 8 = 4$ If $b^M = b^N$, $b > 0$, and $b \neq 1$, then $M = N$.

$3x = 12$ Add 8 to both sides.

$x = 4$ Divide both sides by 3.

Substituting 4 for x into the original equation produces the true statement $16 = 16$. The solution set is $\{4\}$.

b. Because $27 = 3^3$ and $9 = 3^2$, we can express both sides of $27^{x+3} = 9^{x-1}$ in terms of base 3.

$27^{x+3} = 9^{x-1}$ This is the given equation.

$(3^3)^{x+3} = (3^2)^{x-1}$ Write each side as a power of the same base.

$3^{3(x+3)} = 3^{2(x-1)}$ When an exponential expression is raised to a power, multiply exponents.

$3(x + 3) = 2(x - 1)$ If two powers of the same base are equal, then the exponents are equal.

$3x + 9 = 2x - 2$ Apply the distributive property.

$x + 9 = -2$ Subtract $2x$ from both sides.

$x = -11$ Subtract 9 from both sides.

Substituting -11 for x into the original equation produces $27^{-8} = 9^{-12}$, which simplifies to the true statement $3^{-24} = 3^{-24}$. The solution set is $\{-11\}$.

Check Point 1 Solve: **a.** $5^{3x-6} = 125$ **b.** $8^{x+2} = 4^{x-3}$.

2 Use logarithms to solve exponential equations.

Most exponential equations cannot be rewritten so that each side has the same base. Here are two examples:

$4^x = 15$ — We cannot rewrite both sides in terms of base 2 or base 4.

$10^x = 120{,}000.$ — We cannot rewrite both sides in terms of base 10.

Logarithms are extremely useful in solving these equations. The solution begins with isolating the exponential expression. Notice that the exponential expression is already isolated in both $4^x = 15$ and $10^x = 120{,}000$. Then we take the logarithm on both sides. Why can we do this? All logarithmic relations are functions. Thus, if M and N are positive real numbers and $M = N$, then $\log_b M = \log_b N$.

The base that is used when taking the logarithm on both sides of an equation can be any base at all. If the exponential equation involves base 10, as in $10^x = 120{,}000$, we'll take the common logarithm on both sides. If the exponential equation involves any other base, as in $4^x = 15$, we'll take the natural logarithm on both sides.

Using Logarithms to Solve Exponential Equations

1. Isolate the exponential expression.
2. Take the natural logarithm on both sides of the equation for bases other than 10. Take the common logarithm on both sides of the equation for base 10.
3. Simplify using one of the following properties:

$$\ln b^x = x \ln b \quad \text{or} \quad \ln e^x = x \quad \text{or} \quad \log 10^x = x.$$

4. Solve for the variable.

EXAMPLE 2 Solving Exponential Equations

Solve: **a.** $4^x = 15$ **b.** $10^x = 120{,}000$.

Solution We will use the natural logarithmic function to solve $4^x = 15$ and the common logarithmic function to solve $10^x = 120{,}000$.

a. Because the exponential expression, 4^x, is already isolated on the left side of $4^x = 15$, we begin by taking the natural logarithm on both sides of the equation.

$4^x = 15$ This is the given equation.

$\ln 4^x = \ln 15$ Take the natural logarithm on both sides.

$x \ln 4 = \ln 15$ Use the power rule and bring the variable exponent to the front: $\ln b^x = x \ln b$.

$x = \frac{\ln 15}{\ln 4}$ Solve for x by dividing both sides by ln 4.

We now have an exact value for x. We use the exact value for x in the equation's solution set. Thus, the equation's solution is $\frac{\ln 15}{\ln 4}$ and the solution set is $\left\{\frac{\ln 15}{\ln 4}\right\}$. We can obtain a decimal approximation by using a calculator: $x \approx 1.95$. Because $4^2 = 16$, it seems reasonable that the solution to $4^x = 15$ is approximately 1.95.

b. Because the exponential expression, 10^x, is already isolated on the left side of $10^x = 120{,}000$, we begin by taking the common logarithm on both sides of the equation.

$10^x = 120{,}000$ This is the given equation.

$\log 10^x = \log 120{,}000$ Take the common logarithm on both sides.

$x = \log 120{,}000$ Use the inverse property $\log 10^x = x$ on the left.

The equation's solution is log 120,000 and the solution set is {log 120,000}. We can obtain a decimal approximation by using a calculator: $x \approx 5.08$. Because $10^5 = 100{,}000$, it seems reasonable that the solution to $10^x = 120{,}000$ is approximately 5.08.

Discovery

Keep in mind that the base used when taking the logarithm on both sides of an equation can be any base at all. Solve $4^x = 15$ by taking the common logarithm on both sides. Solve again, this time taking the logarithm with base 4 on both sides. Use the change-of-base property to show that the solutions are the same as the one obtained in Example 2(a).

Check Point 2 Solve: **a.** $5^x = 134$ **b.** $10^x = 8000$.

Find each solution set and then use a calculator to obtain a decimal approximation to two decimal places for the solution.

EXAMPLE 3 Solving an Exponential Equation

Solve: $40e^{0.6x} - 3 = 237$.

Solution We begin by adding 3 to both sides and dividing both sides by 40 to isolate the exponential expression, $e^{0.6x}$. Then we take the natural logarithm on both sides of the equation.

$$40e^{0.6x} - 3 = 237 \quad \text{This is the given equation.}$$

$$40e^{0.6x} = 240 \quad \text{Add 3 to both sides.}$$

$$e^{0.6x} = 6 \quad \text{Isolate the exponential factor by dividing both sides by 40.}$$

$$\ln e^{0.6x} = \ln 6 \quad \text{Take the natural logarithm on both sides.}$$

$$0.6x = \ln 6 \quad \text{Use the inverse property } \ln e^x = x \text{ on the left.}$$

$$x = \frac{\ln 6}{0.6} \approx 2.99 \quad \text{Divide both sides by 0.6 and solve for } x.$$

Thus, the solution of the equation is $\frac{\ln 6}{0.6} \approx 2.99$. Try checking this approximate solution in the original equation to verify that $\left\{\frac{\ln 6}{0.6}\right\}$ is the solution set.

Check Point 3 Solve: $7e^{2x} - 5 = 58$. Find the solution set and then use a calculator to obtain a decimal approximation to two decimal places for the solution.

EXAMPLE 4 Solving an Exponential Equation

Solve: $5^{x-2} = 4^{2x+3}$.

Solution Because each exponential expression is isolated on one side of the equation, we begin by taking the natural logarithm on both sides.

$$5^{x-2} = 4^{2x+3} \quad \text{This is the given equation.}$$

$$\ln 5^{x-2} = \ln 4^{2x+3} \quad \text{Take the natural logarithm on both sides.}$$

Be sure to insert parentheses around the binomials.

$$(x-2)\ln 5 = (2x+3)\ln 4 \quad \text{Use the power rule and bring the variable exponents to the front: } \ln b^x = x \ln b.$$

Remember that ln 5 and ln 4 are constants, not variables.

$$x\ln 5 - 2\ln 5 = 2x\ln 4 + 3\ln 4 \quad \text{Use the distributive property to distribute ln 5 and ln 4 to both terms in parentheses.}$$

$$x\ln 5 - 2x\ln 4 = 2\ln 5 + 3\ln 4 \quad \text{Collect variable terms involving } x \text{ on the left by subtracting } 2x\ln 4 \text{ and adding } 2\ln 5 \text{ on both sides.}$$

$$x(\ln 5 - 2\ln 4) = 2\ln 5 + 3\ln 4 \quad \text{Factor out } x \text{ from the two terms on the left.}$$

$$x = \frac{2\ln 5 + 3\ln 4}{\ln 5 - 2\ln 4} \quad \text{Isolate } x \text{ by dividing both sides by } \ln 5 - 2\ln 4.$$

The solution set is $\left\{\frac{2\ln 5 + 3\ln 4}{\ln 5 - 2\ln 4}\right\}$. The solution is approximately -6.34.

Discovery

Use properties of logarithms to show that the solution in Example 4 can be expressed as

$$\frac{\ln 1600}{\ln\left(\frac{5}{16}\right)}.$$

Check Point 4 Solve: $3^{2x-1} = 7^{x+1}$. Find the solution set and then use a calculator to obtain a decimal approximation to two decimal places for the solution.

EXAMPLE 5 Solving an Exponential Equation

Solve: $e^{2x} - 4e^{x} + 3 = 0.$

Solution The given equation is quadratic in form. If $u = e^{x}$, the equation can be expressed as $u^2 - 4u + 3 = 0$. Because this equation can be solved by factoring, we factor to isolate the exponential term.

$$e^{2x} - 4e^{x} + 3 = 0 \quad \text{This is the given equation.}$$

$$(e^{x} - 3)(e^{x} - 1) = 0 \quad \text{Factor on the left. Notice that if } u = e^{x}, u^2 - 4u + 3 = (u - 3)(u - 1).$$

$$e^{x} - 3 = 0 \quad \text{or} \quad e^{x} - 1 = 0 \quad \text{Set each factor equal to 0.}$$

$$e^{x} = 3 \qquad e^{x} = 1 \quad \text{Solve for } e^{x}.$$

$$\ln e^{x} = \ln 3 \qquad x = 0 \quad \text{Take the natural logarithm on both sides of the first equation. The equation on the right can be solved by inspection.}$$

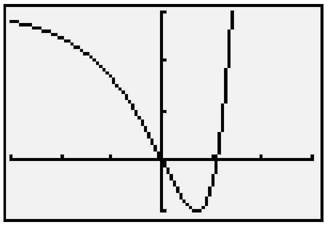
$$x = \ln 3 \quad \ln e^{x} = x$$

The solution set is $\{0, \ln 3\}$. The solutions are 0 and ln 3, which is approximately 1.10.

Technology

Graphic Connections

Shown below is the graph of $y = e^{2x} - 4e^{x} + 3$. There are two x-intercepts, one at 0 and one at approximately 1.10. These intercepts verify our algebraic solution.



$[-3, 3, 1]$ by $[-1, 3, 1]$

Check Point 5 Solve: $e^{2x} - 8e^{x} + 7 = 0$. Find the solution set and then use a calculator to obtain a decimal approximation to two decimal places, if necessary, for the solutions.

3 Use the definition of a logarithm to solve logarithmic equations.

Logarithmic Equations

A **logarithmic equation** is an equation containing a variable in a logarithmic expression. Examples of logarithmic equations include

$$\log_4(x + 3) = 2 \quad \text{and} \quad \ln(x + 2) - \ln(4x + 3) = \ln\left(\frac{1}{x}\right).$$

Some logarithmic equations can be expressed in the form $\log_b M = c$. We can solve such equations by rewriting them in exponential form.

Using the Definition of a Logarithm to Solve Logarithmic Equations

1. Express the equation in the form $\log_b M = c$.
2. Use the definition of a logarithm to rewrite the equation in exponential form:

$$\log_b M = c \quad \text{means} \quad b^{c} = M.$$

Logarithms are exponents.

3. Solve for the variable.
4. Check proposed solutions in the original equation. Include in the solution set only values for which $M > 0$.

EXAMPLE 6 Solving Logarithmic Equations

Solve: **a.** $\log_4(x + 3) = 2$ **b.** $3\ln(2x) = 12$.

Solution The form $\log_b M = c$ involves a single logarithm whose coefficient is 1 on one side and a constant on the other side. Equation (a) is already in this form. We will need to divide both sides of equation (b) by 3 to obtain this form.

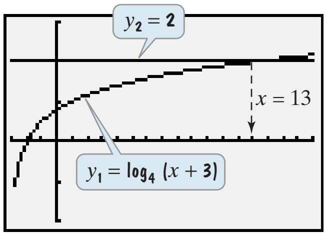
Technology

Graphic Connections

The graphs of

$$y_1 = \log_4(x+3) \text{ and } y_2 = 2$$

have an intersection point whose x-coordinate is 13. This verifies that $\{13\}$ is the solution set for $\log_4(x+3) = 2$.



$[-3, 17, 1]$ by $[-2, 3, 1]$

a. $\log_4(x+3) = 2$ — This is the given equation.

$4^2 = x + 3$ — Rewrite in exponential form: $\log_b M = c$ means $b^c = M$.

$16 = x + 3$ — Square 4.

$13 = x$ — Subtract 3 from both sides.

Check 13:

$\log_4(x+3) = 2$ — This is the given logarithmic equation.

$\log_4(13+3) \stackrel{?}{=} 2$ — Substitute 13 for x.

$\log_4 16 \stackrel{?}{=} 2$

$2 = 2$, true — $\log_4 16 = 2$ because $4^2 = 16$.

This true statement indicates that the solution set is $\{13\}$.

b. $3\ln(2x) = 12$ — This is the given equation.

$\ln(2x) = 4$ — Divide both sides by 3.

$\log_e(2x) = 4$ — Rewrite the natural logarithm showing base e. This step is optional.

$e^4 = 2x$ — Rewrite in exponential form: $\log_b M = c$ means $b^c = M$.

$\frac{e^4}{2} = x$ — Divide both sides by 2.

Check $\frac{e^4}{2}$:

$3\ln(2x) = 12$ — This is the given logarithmic equation.

$3\ln\left[2\left(\frac{e^4}{2}\right)\right] \stackrel{?}{=} 12$ — Substitute $\frac{e^4}{2}$ for x.

$3\ln e^4 \stackrel{?}{=} 12$ — Simplify: $\frac{\cancel{2}}{1} \cdot \frac{e^4}{\cancel{2}} = e^4$.

$3 \cdot 4 \stackrel{?}{=} 12$ — Because $\ln e^x = x$, we conclude $\ln e^4 = 4$.

$12 = 12$, true

This true statement indicates that the solution set is $\left\{\frac{e^4}{2}\right\}$.

Check Point 6

Solve: **a.** $\log_2(x-4) = 3$ **b.** $4\ln(3x) = 8$.

Logarithmic expressions are defined only for logarithms of positive real numbers. **Always check proposed solutions of a logarithmic equation in the original equation. Exclude from the solution set any proposed solution that produces the logarithm of a negative number or the logarithm of 0.**

To rewrite the logarithmic equation $\log_b M = c$ in the equivalent exponential form $b^c = M$, we need a single logarithm whose coefficient is one. It is sometimes necessary to use properties of logarithms to condense logarithms into a single logarithm. In the next example, we use the product rule for logarithms to obtain a single logarithmic expression on the left side.

EXAMPLE 7 Solving a Logarithmic Equation

Solve: $\log_2 x + \log_2(x-7) = 3$.

Solution

$\log_2 x + \log_2(x-7) = 3$ — This is the given equation.

$\log_2[x(x-7)] = 3$ — Use the product rule to obtain a single logarithm: $\log_b M + \log_b N = \log_b(MN)$.

$2^3 = x(x-7)$	Rewrite in exponential form: $\log_b M = c$ means $b^c = M$.
$8 = x^2 - 7x$	Evaluate 2^3 on the left and apply the distributive property on the right.
$0 = x^2 - 7x - 8$	Set the equation equal to 0.
$0 = (x-8)(x+1)$	Factor.
$x - 8 = 0$ or $x + 1 = 0$	Set each factor equal to 0.
$x = 8$ $\quad$ $x = -1$	Solve for x.

Check 8:	Check −1:
$\log_2 x + \log_2(x-7) = 3$	$\log_2 x + \log_2(x-7) = 3$
$\log_2 8 + \log_2(8-7) \stackrel{?}{=} 3$	$\log_2(-1) + \log_2(-1-7) \stackrel{?}{=} 3$
$\log_2 8 + \log_2 1 \stackrel{?}{=} 3$	The number −1 does not check.
$3 + 0 \stackrel{?}{=} 3$	Negative numbers do not have logarithms.
$3 = 3$, true	

The solution set is $\{8\}$.

Check Point 7 Solve: $\log x + \log(x-3) = 1$.

4 Use the one-to-one property of logarithms to solve logarithmic equations.

Some logarithmic equations can be expressed in the form $\log_b M = \log_b N$, where the bases on both sides of the equation are the same. Because all logarithmic functions are one-to-one, we can conclude that $M = N$.

Using the One-to-One Property of Logarithms to Solve Logarithmic Equations

1. Express the equation in the form $\log_b M = \log_b N$. This form involves a single logarithm whose coefficient is 1 on each side of the equation.
2. Use the one-to-one property to rewrite the equation without logarithms: If $\log_b M = \log_b N$, then $M = N$.
3. Solve for the variable.
4. Check proposed solutions in the original equation. Include in the solution set only values for which $M > 0$ and $N > 0$.

EXAMPLE 8 Solving a Logarithmic Equation

Solve: $\ln(x+2) - \ln(4x+3) = \ln\left(\frac{1}{x}\right)$.

Solution In order to apply the one-to-one property of logarithms, we need a single logarithm whose coefficient is 1 on each side of the equation. The right side is already in this form. We can obtain a single logarithm on the left side by applying the quotient rule.

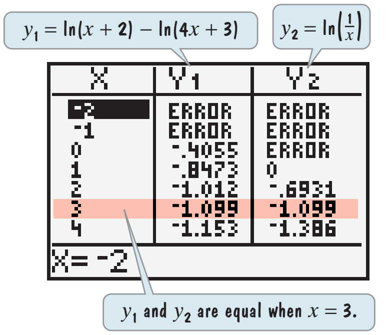
$\ln(x+2) - \ln(4x+3) = \ln\left(\frac{1}{x}\right)$	This is the given equation.
$\ln\left(\frac{x+2}{4x+3}\right) = \ln\left(\frac{1}{x}\right)$	Use the quotient rule to obtain a single logarithm on the left side: $\log_b M - \log_b N = \log_b\left(\frac{M}{N}\right)$.
$\frac{x+2}{4x+3} = \frac{1}{x}$	Use the one-to-one property: If $\log_b M = \log_b N$, then $M = N$.

Technology

Numeric Connections

A graphing utility's TABLE feature can be used to verify that $\{3\}$ is the solution set of

$$\ln(x+2) - \ln(4x+3) = \ln\left(\frac{1}{x}\right).$$



$$x(4x+3)\left(\frac{x+2}{4x+3}\right) = x(4x+3)\left(\frac{1}{x}\right) \quad \text{Multiply both sides by } x(4x+3)\text{, the LCD.}$$

$$x(x+2) = 4x+3 \quad \text{Simplify.}$$

$$x^2 + 2x = 4x + 3 \quad \text{Apply the distributive property.}$$

$$x^2 - 2x - 3 = 0 \quad \text{Subtract } 4x+3 \text{ from both sides and set the equation equal to 0.}$$

$$(x-3)(x+1) = 0 \quad \text{Factor.}$$

$$x - 3 = 0 \quad \text{or} \quad x + 1 = 0 \quad \text{Set each factor equal to 0.}$$

$$x = 3 \qquad x = -1 \quad \text{Solve for } x.$$

Substituting 3 for x into $\ln(x+2) - \ln(4x+3) = \ln\left(\frac{1}{x}\right)$ produces the true statement $\ln\left(\frac{1}{3}\right) = \ln\left(\frac{1}{3}\right)$. However, substituting -1 produces logarithms of negative numbers. Thus, -1 is not a solution. The solution set is $\{3\}$.

Check Point 8 Solve: $\ln(x-3) = \ln(7x-23) - \ln(x+1)$.

Applications

5 Solve applied problems involving exponential and logarithmic equations.

Our first applied example provides a mathematical perspective on the old slogan "Alcohol and driving don't mix." In California, where 38% of fatal traffic crashes involve drinking drivers, it is illegal to drive with a blood alcohol concentration of 0.08 or higher. At these levels, drivers may be arrested and charged with driving under the influence.

EXAMPLE 9 Alcohol and Risk of a Car Accident

Medical research indicates that the risk of having a car accident increases exponentially as the concentration of alcohol in the blood increases. The risk is modeled by

$$R = 6e^{12.77x},$$

where x is the blood alcohol concentration and R, given as a percent, is the risk of having a car accident. What blood alcohol concentration corresponds to a 17% risk of a car accident? How is this shown on the graph of R in **Figure 3.20**?

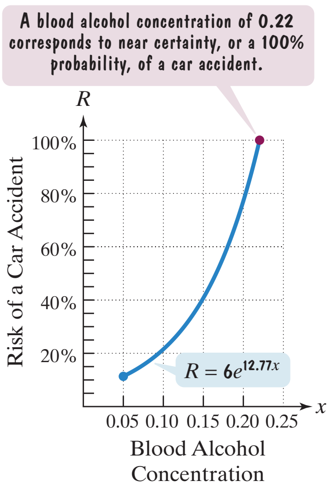


Figure 3.20

Solution For a risk of 17%, we let $R = 17$ in the equation and solve for x, the blood alcohol concentration.

$$R = 6e^{12.77x} \quad \text{This is the given equation.}$$

$$6e^{12.77x} = 17 \quad \text{Substitute 17 for } R \text{ and (optional) reverse the two sides of the equation.}$$

$$e^{12.77x} = \frac{17}{6} \quad \text{Isolate the exponential factor by dividing both sides by 6.}$$

$$\ln e^{12.77x} = \ln\left(\frac{17}{6}\right) \quad \text{Take the natural logarithm on both sides.}$$

$$12.77x = \ln\left(\frac{17}{6}\right) \quad \text{Use the inverse property } \ln e^x = x \text{ on the left side.}$$

$$x = \frac{\ln\left(\frac{17}{6}\right)}{12.77} \approx 0.08 \quad \text{Divide both sides by 12.77.}$$

For a blood alcohol concentration of 0.08, the risk of a car accident is 17%. This is shown on the graph of R in **Figure 3.20** by the point (0.08, 17) that lies on the blue curve. Take a moment to locate this point on the curve. In many states, it is illegal to drive with a blood alcohol concentration of 0.08.

Check Point 9 Use the formula in Example 9 to answer this question: What blood alcohol concentration corresponds to a 7% risk of a car accident? (In many states, drivers under the age of 21 can lose their licenses for driving at this level.)

Suppose that you inherit \$30,000 at age 20. Is it possible to invest \$25,000 and have over half a million dollars for early retirement? Our next example illustrates the power of compound interest.

EXAMPLE 10 Revisiting the Formula for Compound Interest

The formula

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

describes the accumulated value, A, of a sum of money, P, the principal, after t years at annual percentage rate r (in decimal form) compounded n times a year. How long will it take \$25,000 to grow to \$500,000 at 9% annual interest compounded monthly?

Solution

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$ This is the given formula.

$$500{,}000 = 25{,}000\left(1 + \frac{0.09}{12}\right)^{12t}$$ A(the desired accumulated value) = \$500,000, P(the principal) = \$25,000, r(the interest rate) = 9% = 0.09, and $n = 12$ (monthly compounding).

Because of the exponent, $12t$, do not distribute 25,000 over the terms in parentheses.

Our goal is to solve the equation for t. Let's reverse the two sides of the equation and then simplify within parentheses.

$$25{,}000\left(1 + \frac{0.09}{12}\right)^{12t} = 500{,}000$$ Reverse the two sides of the previous equation.

$$25{,}000(1 + 0.0075)^{12t} = 500{,}000$$ Divide within parentheses: $\frac{0.09}{12} = 0.0075$.

$$25{,}000(1.0075)^{12t} = 500{,}000$$ Add within parentheses.

$$(1.0075)^{12t} = 20$$ Divide both sides by 25,000.

$$\ln(1.0075)^{12t} = \ln 20$$ Take the natural logarithm on both sides.

$$12t \ln(1.0075) = \ln 20$$ Use the power rule to bring the exponent to the front: $\ln b^x = x \ln b$.

$$t = \frac{\ln 20}{12 \ln 1.0075}$$ Solve for t, dividing both sides by 12 ln 1.0075.

$$\approx 33.4$$ Use a calculator.

After approximately 33.4 years, the \$25,000 will grow to an accumulated value of \$500,000. If you set aside the money at age 20, you can begin enjoying a life of leisure at about age 53.

Check Point 10 How long, to the nearest tenth of a year, will it take \$1000 to grow to \$3600 at 8% annual interest compounded quarterly?

Playing Doubles: Interest Rates and Doubling Time

One way to calculate what your savings will be worth at some point in the future is to consider doubling time. The following table shows how long it takes for your money to double at different annual interest rates subject to continuous compounding.

Annual Interest Rate	Years to Double
5%	13.9 years
7%	9.9 years
9%	7.7 years
11%	6.3 years

Of course, the first problem is collecting some money to invest. The second problem is finding a reasonably safe investment with a return of 9% or more.

EXAMPLE 11 Fair or Poor Health, by Annual Income

The bar graph in **Figure 3.21** shows people with lower incomes are more likely to report that their health is fair or poor. The function

$$f(x) = 54.8 - 12.3 \ln x$$

models the percentage of Americans reporting fair or poor health, $f(x)$, in terms of annual income, x, in thousands of dollars. According to the model, what annual income corresponds to 10% reporting fair or poor health? Round to the nearest thousand dollars.

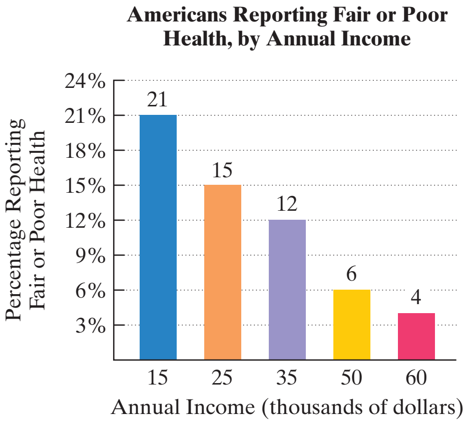


Figure 3.21
Source: William Kornblum and Joseph Julian, *Social Problems*, Twelfth Edition, Prentice Hall, 2007

Solution To find what annual income corresponds to 10% reporting fair or poor health, we substitute 10 for $f(x)$ and solve for x, the annual income.

$f(x) = 54.8 - 12.3 \ln x$ — This is the given function.

$10 = 54.8 - 12.3 \ln x$ — Substitute 10 for f(x).

Our goal is to isolate ln x and then rewrite the equation in exponential form.

$-44.8 = -12.3 \ln x$ — Subtract 54.8 from both sides.

$\frac{44.8}{12.3} = \ln x$ — Divide both sides by −12.3.

$\frac{44.8}{12.3} = \log_e x$ — Rewrite the natural logarithm showing base e. This step is optional.

$e^{\frac{44.8}{12.3}} = x$ — Rewrite in exponential form.

$38 \approx x$ — Use a calculator.

An annual income of approximately \$38,000 corresponds to 10% of Americans reporting fair or poor health.

Check Point 11 According to the function in Example 11, what annual income corresponds to 25% reporting fair or poor health? Round to the nearest thousand dollars.

Exercise Set 3.4

Practice Exercises

Solve each exponential equation in Exercises 1–22 by expressing each side as a power of the same base and then equating exponents.

1. $2^x = 64$
2. $3^x = 81$
3. $5^x = 125$
4. $5^x = 625$
5. $2^{2x-1} = 32$
6. $3^{2x+1} = 27$
7. $4^{2x-1} = 64$
8. $5^{3x-1} = 125$
9. $32^x = 8$
10. $4^x = 32$
11. $9^x = 27$
12. $125^x = 625$
13. $3^{1-x} = \frac{1}{27}$
14. $5^{2-x} = \frac{1}{125}$
15. $6^{\frac{x-3}{4}} = \sqrt{6}$
16. $7^{\frac{x-2}{6}} = \sqrt{7}$
17. $4^x = \frac{1}{\sqrt{2}}$
18. $9^x = \frac{1}{\sqrt[3]{3}}$
19. $8^{x+3} = 16^{x-1}$
20. $8^{1-x} = 4^{x+2}$
21. $e^{x+1} = \frac{1}{e}$
22. $e^{x+4} = \frac{1}{e^{2x}}$

Solve each exponential equation in Exercises 23–48. Express the solution set in terms of natural logarithms. Then use a calculator to obtain a decimal approximation, correct to two decimal places, for the solution.

23. $10^x = 3.91$

24. $10^x = 8.07$

25. $e^x = 5.7$

26. $e^x = 0.83$

27. $5^x = 17$

28. $19^x = 143$

29. $5e^x = 23$

30. $9e^x = 107$

31. $3e^{5x} = 1977$

32. $4e^{7x} = 10{,}273$

33. $e^{1-5x} = 793$

34. $e^{1-8x} = 7957$

35. $e^{5x-3} - 2 = 10{,}476$

36. $e^{4x-5} - 7 = 11{,}243$

37. $7^{x+2} = 410$

38. $5^{x-3} = 137$

39. $7^{0.3x} = 813$

40. $3^{\frac{x}{7}} = 0.2$

41. $5^{2x+3} = 3^{x-1}$

42. $7^{2x+1} = 3^{x+2}$

43. $e^{2x} - 3e^x + 2 = 0$

44. $e^{2x} - 2e^x - 3 = 0$

45. $e^{4x} + 5e^{2x} - 24 = 0$

46. $e^{4x} - 3e^{2x} - 18 = 0$

47. $3^{2x} + 3^x - 2 = 0$

48. $2^{2x} + 2^x - 12 = 0$

Solve each logarithmic equation in Exercises 49–90. Be sure to reject any value of x that is not in the domain of the original logarithmic expressions. Give the exact answer. Then, where necessary, use a calculator to obtain a decimal approximation, correct to two decimal places, for the solution.

49. $\log_3 x = 4$

50. $\log_5 x = 3$

51. $\ln x = 2$

52. $\ln x = 3$

53. $\log_4(x + 5) = 3$

54. $\log_5(x - 7) = 2$

55. $\log_3(x - 4) = -3$

56. $\log_7(x + 2) = -2$

57. $\log_4(3x + 2) = 3$

58. $\log_2(4x + 1) = 5$

59. $5 \ln(2x) = 20$

60. $6 \ln(2x) = 30$

61. $6 + 2 \ln x = 5$

62. $7 + 3 \ln x = 6$

63. $\ln \sqrt{x + 3} = 1$

64. $\ln \sqrt{x + 4} = 1$

65. $\log_5 x + \log_5(4x - 1) = 1$

66. $\log_6(x + 5) + \log_6 x = 2$

67. $\log_3(x - 5) + \log_3(x + 3) = 2$

68. $\log_2(x - 1) + \log_2(x + 1) = 3$

69. $\log_2(x + 2) - \log_2(x - 5) = 3$

70. $\log_4(x + 2) - \log_4(x - 1) = 1$

71. $2 \log_3(x + 4) = \log_3 9 + 2$

72. $3 \log_2(x - 1) = 5 - \log_2 4$

73. $\log_2(x - 6) + \log_2(x - 4) - \log_2 x = 2$

74. $\log_2(x - 3) + \log_2 x - \log_2(x + 2) = 2$

75. $\log(x + 4) = \log x + \log 4$

76. $\log(5x + 1) = \log(2x + 3) + \log 2$

77. $\log(3x - 3) = \log(x + 1) + \log 4$

78. $\log(2x - 1) = \log(x + 3) + \log 3$

79. $2 \log x = \log 25$

80. $3 \log x = \log 125$

81. $\log(x + 4) - \log 2 = \log(5x + 1)$

82. $\log(x + 7) - \log 3 = \log(7x + 1)$

83. $2 \log x - \log 7 = \log 112$

84. $\log(x - 2) + \log 5 = \log 100$

85. $\log x + \log(x + 3) = \log 10$

86. $\log(x + 3) + \log(x - 2) = \log 14$

87. $\ln(x - 4) + \ln(x + 1) = \ln(x - 8)$

88. $\log_2(x - 1) - \log_2(x + 3) = \log_2\left(\frac{1}{x}\right)$

89. $\ln(x - 2) - \ln(x + 3) = \ln(x - 1) - \ln(x + 7)$

90. $\ln(x - 5) - \ln(x + 4) = \ln(x - 1) - \ln(x + 2)$

Practice Plus

In Exercises 91–100, solve each equation.

91. $5^{2x} \cdot 5^{4x} = 125$

92. $3^{x+2} \cdot 3^x = 81$

93. $2\,|\ln x| - 6 = 0$

94. $3\,|\log x| - 6 = 0$

95. $3^{x^2} = 45$

96. $5^{x^2} = 50$

97. $\ln(2x + 1) + \ln(x - 3) - 2 \ln x = 0$

98. $\ln 3 - \ln(x + 5) - \ln x = 0$

99. $5^{x^2-12} = 25^{2x}$

100. $3^{x^2-12} = 9^{2x}$

Application Exercises

101. The formula $A = 36.1e^{0.0126t}$ models the population of California, A, in millions, t years after 2005.

a. What was the population of California in 2005?

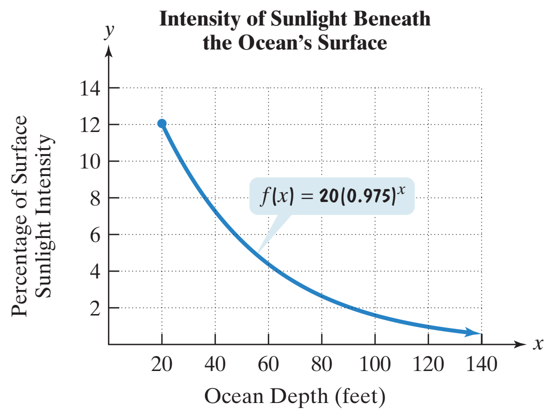
b. When will the population of California reach 40 million?

102. The formula $A = 22.9e^{0.0183t}$ models the population of Texas, A, in millions, t years after 2005.

a. What was the population of Texas in 2005?

b. When will the population of Texas reach 27 million?

The function $f(x) = 20(0.975)^x$ models the percentage of surface sunlight, $f(x)$, that reaches a depth of x feet beneath the surface of the ocean. The figure shows the graph of this function. Use this information to solve Exercises 103–104.



103. Use the function to determine at what depth, to the nearest foot, there is 1% of surface sunlight. How is this shown on the graph of f?

104. Use the function to determine at what depth, to the nearest foot, there is 3% of surface sunlight. How is this shown on the graph of f?

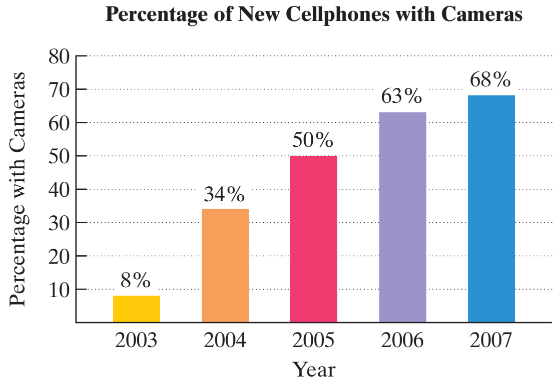
In Exercises 105–108, complete the table for a savings account subject to n compoundings yearly $\left[A = P\left(1 + \frac{r}{n}\right)^{nt}\right]$. *Round answers to one decimal place.*

	Amount Invested	Number of Compounding Periods	Annual Interest Rate	Accumulated Amount	Time t in Years
105.	\$12,500	4	5.75%	\$20,000	
106.	\$7250	12	6.5%	\$15,000	
107.	\$1000	360		\$1400	2
108.	\$5000	360		\$9000	4

In Exercises 109–112, complete the table for a savings account subject to continuous compounding $(A = Pe^{rt})$. *Round answers to one decimal place.*

	Amount Invested	Annual Interest Rate	Accumulated Amount	Time t in Years
109.	\$8000	8%	Double the amount invested	
110.	\$8000		\$12,000	2
111.	\$2350		Triple the amount invested	7
112.	\$17,425	4.25%	\$25,000	

The days when something happens and it's not captured on camera appear to be over. The bar graph shows that by 2007, 68% of cellphones sold in the United States were equipped with cameras.



Source: Strategy Analytics Inc.

The data can be modeled by the function $f(x) = 8 + 38 \ln x$, *where* $f(x)$ *is the percentage of new cellphones with cameras* x *years after 2002. Use this information to solve Exercises 113–114.*

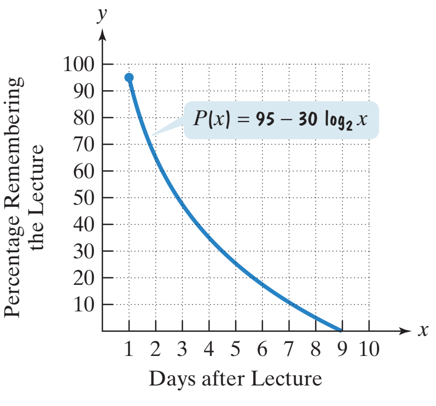
113. a. Use the function to determine the percentage of new cellphones with cameras in 2007. Round to the nearest whole percent. Does this overestimate or underestimate the percent displayed by the graph? By how much?

b. If trends shown from 2003 through 2007 continue, use the function to determine by which year 87% of new cellphones will be equipped with cameras.

114. a. Use the function to determine the percentage of new cellphones with cameras in 2006. Round to the nearest whole percent. Does this overestimate or underestimate the percent displayed by the graph? By how much?

b. If trends shown from 2003 through 2007 continue, use the function to determine by which year all new cellphones will be equipped with cameras.

The function $P(x) = 95 - 30 \log_2 x$ *models the percentage,* $P(x)$, *of students who could recall the important features of a classroom lecture as a function of time, where* x *represents the number of days that have elapsed since the lecture was given. The figure shows the graph of the function. Use this information to solve Exercises 115–116. Round answers to one decimal place.*

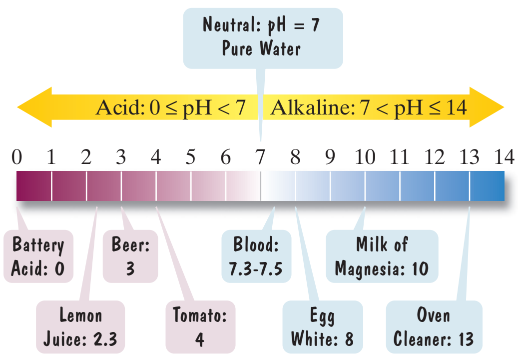


115. After how many days do only half the students recall the important features of the classroom lecture? (Let $P(x) = 50$ and solve for x.) Locate the point on the graph that conveys this information.

116. After how many days have all students forgotten the important features of the classroom lecture? (Let $P(x) = 0$ and solve for x.) Locate the point on the graph that conveys this information.

The pH scale is used to measure the acidity or alkalinity of a solution. The scale ranges from 0 to 14. A neutral solution, such as pure water, has a pH of 7. An acid solution has a pH less than 7 and an alkaline solution has a pH greater than 7. The lower the pH below 7, the more acidic is the solution. Each whole-number decrease in pH represents a tenfold increase in acidity.

The pH Scale



The pH of a solution is given by

$$\text{pH} = -\log x,$$

where x represents the concentration of the hydrogen ions in the solution, in moles per liter. Use the formula to solve Exercises 117–118. Express answers as powers of 10.

117. a. Normal, unpolluted rain has a pH of about 5.6. What is the hydrogen ion concentration?

b. An environmental concern involves the destructive effects of acid rain. The most acidic rainfall ever had a pH of 2.4. What was the hydrogen ion concentration?

c. How many times greater is the hydrogen ion concentration of the acidic rainfall in part (b) than the normal rainfall in part (a)?

118. a. The figure indicates that lemon juice has a pH of 2.3. What is the hydrogen ion concentration?

b. Stomach acid has a pH that ranges from 1 to 3. What is the hydrogen ion concentration of the most acidic stomach?

c. How many times greater is the hydrogen ion concentration of the acidic stomach in part (b) than the lemon juice in part (a)?

Writing in Mathematics

119. Explain how to solve an exponential equation when both sides can be written as a power of the same base.

120. Explain how to solve an exponential equation when both sides cannot be written as a power of the same base. Use $3^x = 140$ in your explanation.

121. Explain the differences between solving $\log_3(x-1) = 4$ and $\log_3(x-1) = \log_3 4$.

122. In many states, a 17% risk of a car accident with a blood alcohol concentration of 0.08 is the lowest level for charging a motorist with driving under the influence. Do you agree with the 17% risk as a cutoff percentage, or do you feel that the percentage should be lower or higher? Explain your answer. What blood alcohol concentration corresponds to what you believe is an appropriate percentage?

Technology Exercises

In Exercises 123–130, use your graphing utility to graph each side of the equation in the same viewing rectangle. Then use the x-coordinate of the intersection point to find the equation's solution set. Verify this value by direct substitution into the equation.

123. $2^{x+1} = 8$

124. $3^{x+1} = 9$

125. $\log_3(4x-7) = 2$

126. $\log_3(3x-2) = 2$

127. $\log(x+3) + \log x = 1$

128. $\log(x-15) + \log x = 2$

129. $3^x = 2x + 3$

130. $5^x = 3x + 4$

Hurricanes are one of nature's most destructive forces. These low-pressure areas often have diameters of over 500 miles. The function $f(x) = 0.48\ln(x+1) + 27$ models the barometric air pressure, $f(x)$, in inches of mercury, at a distance of x miles from the eye of a hurricane. Use this function to solve Exercises 131–132.

131. Graph the function in a [0, 500, 50] by [27, 30, 1] viewing rectangle. What does the shape of the graph indicate about barometric air pressure as the distance from the eye increases?

132. Use an equation to answer this question: How far from the eye of a hurricane is the barometric air pressure 29 inches of mercury? Use the TRACE and ZOOM features or the intersect command of your graphing utility to verify your answer.

133. The function $P(t) = 145e^{-0.092t}$ models a runner's pulse, $P(t)$, in beats per minute, t minutes after a race, where $0 \le t \le 15$. Graph the function using a graphing utility. TRACE along the graph and determine after how many minutes the runner's pulse will be 70 beats per minute. Round to the nearest tenth of a minute. Verify your observation algebraically.

134. The function $W(t) = 2600(1 - 0.51e^{-0.075t})^3$ models the weight, $W(t)$, in kilograms, of a female African elephant at age t years. (1 kilogram ≈ 2.2 pounds) Use a graphing utility to graph the function. Then TRACE along the curve to estimate the age of an adult female elephant weighing 1800 kilograms.

Critical Thinking Exercises

Make Sense? *In Exercises 135–138, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

135. Because the equations $2^x = 15$ and $2^x = 16$ are similar, I solved them using the same method.

136. Because the equations

$$\log(3x+1) = 5 \text{ and } \log(3x+1) = \log 5$$

are similar, I solved them using the same method.

137. I can solve $4^x = 15$ by writing the equation in logarithmic form.

138. It's important for me to check that the proposed solution of an equation with logarithms gives only logarithms of positive numbers in the original equation.

In Exercises 139–142, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

139. If $\log(x+3) = 2$, then $e^2 = x + 3$.

140. If $\log(7x + 3) - \log(2x + 5) = 4$, then the equation in exponential form is $10^4 = (7x + 3) - (2x + 5)$.

141. If $x = \frac{1}{k}\ln y$, then $y = e^{kx}$.

142. Examples of exponential equations include $10^x = 5.71$, $e^x = 0.72$, and $x^{10} = 5.71$.

143. If \$4000 is deposited into an account paying 3% interest compounded annually and at the same time \$2000 is deposited into an account paying 5% interest compounded annually, after how long will the two accounts have the same balance? Round to the nearest year.

Solve each equation in Exercises 144–146. Check each proposed solution by direct substitution or with a graphing utility.

144. $(\ln x)^2 = \ln x^2$

145. $(\log x)(2\log x + 1) = 6$

146. $\ln(\ln x) = 0$

Group Exercise

147. Research applications of logarithmic functions as mathematical models and plan a seminar based on your group's research. Each group member should research one of the following areas or any other area of interest: pH (acidity of solutions), intensity of sound (decibels), brightness of stars, human memory, progress over time in a sport, profit over time. For the area that you select, explain how logarithmic functions are used and provide examples.

Preview Exercises

Exercises 148–150 will help you prepare for the material covered in the next section.

148. The formula $A = 10e^{-0.003t}$ models the population of Hungary, A, in millions, t years after 2006.

a. Find Hungary's population, in millions, for 2006, 2007, 2008, and 2009. Round to two decimal places.

b. Is Hungary's population increasing or decreasing?

149. The table shows the average amount that Americans paid for a gallon of gasoline from 2002 through 2006. Create a scatter plot for the data. Based on the shape of the scatter plot, would a logarithmic function, an exponential function, or a linear function be the best choice for modeling the data?

Average Gas Price in the U.S.

Year	Average Price per Gallon
2002	\$1.40
2003	\$1.60
2004	\$1.92
2005	\$2.30
2006	\$2.91

Source: Oil Price Information Service

150. **a.** Simplify: $e^{\ln 3}$.

b. Use your simplification from part (a) to rewrite 3^x in terms of base e.

Section 3.5 Exponential Growth and Decay; Modeling Data

Objectives

1. Model exponential growth and decay.
2. Use logistic growth models.
3. Use Newton's Law of Cooling.
4. Choose an appropriate model for data.
5. Express an exponential model in base e.

The most casual cruise on the Internet shows how people disagree when it comes to making predictions about the effects of the world's growing population. Some argue that there is a recent slowdown in the growth rate, economies remain robust, and famines in North Korea and Ethiopia are aberrations rather than signs of the future. Others say that the 6.8 billion people on Earth is twice as many as can be supported in middle-class comfort, and the world is running out of arable land and fresh water. Debates about entities that are growing exponentially can be approached mathematically: We can create functions that model data and use these functions to make predictions. In this section, we will show you how this is done.

1 Model exponential growth and decay.

Exponential Growth and Decay

One of algebra's many applications is to predict the behavior of variables. This can be done with *exponential growth* and *decay models*. With exponential growth or decay, quantities grow or decay at a rate directly proportional to their size.