Section 2.7 Polynomial and Rational Inequalities

Objectives

1. Solve polynomial inequalities.
2. Solve rational inequalities.
3. Solve problems modeled by polynomial or rational inequalities.

Tailgaters beware: If your car is going 35 miles per hour on dry pavement, your required stopping distance is 160 feet, or the width of a football field. At 65 miles per hour, the distance required is 410 feet, or approximately the length of one and one-tenth football fields. **Figure 2.43** shows stopping distances for cars at various speeds on dry roads and on wet roads.

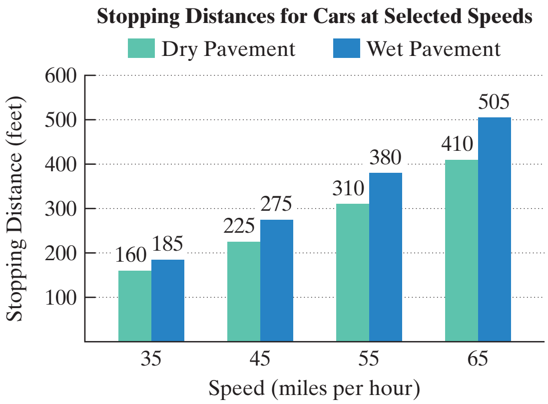


Figure 2.43

Source: National Highway Traffic Safety Administration

A car's required stopping distance, $f(x)$, in feet, on dry pavement traveling at x miles per hour can be modeled by the quadratic function

$$f(x) = 0.0875x^2 - 0.4x + 66.6.$$

How can we use this function to determine speeds on dry pavement requiring stopping distances that exceed the length of one and one-half football fields, or 540 feet? We must solve the inequality

$$\underbrace{0.0875x^2 - 0.4x + 66.6}_{\text{Required stopping distance}} \underbrace{>}_{\text{exceeds}} \underbrace{540.}_{\text{540 feet.}}$$

We begin by subtracting 540 from both sides. This will give us zero on the right:

$$0.0875x^2 - 0.4x + 66.6 - 540 > 540 - 540$$
$$0.0875x^2 - 0.4x - 473.4 > 0.$$

Technology

We used the statistical menu of a graphing utility and the quadratic regression program to obtain the quadratic function that models stopping distance on dry pavement. After entering the appropriate data from **Figure 2.43**, namely

(35, 160), (45, 225), (55, 310), (65, 410),

we obtained the results shown in the screen.

```
QuadReg
 y=ax²+bx+c
 a=.0875
 b=-.4
 c=66.5625
```

The form of this inequality is $ax^2 + bx + c > 0$. Such a quadratic inequality is called a *polynomial inequality*.

Definition of a Polynomial Inequality

A polynomial inequality is any inequality that can be put into one of the forms

$$f(x) < 0, \quad f(x) > 0, \quad f(x) \le 0, \quad \text{or} \quad f(x) \ge 0,$$

where f is a polynomial function.

In this section, we establish the basic techniques for solving polynomial inequalities. We will also use these techniques to solve inequalities involving rational functions.

Solving Polynomial Inequalities

Graphs can help us visualize the solutions of polynomial inequalities. For example, the graph of $f(x) = x^2 - 7x + 10$ is shown in **Figure 2.44**. The x-intercepts, 2 and 5, are **boundary points** between where the graph lies above the x-axis, shown in blue, and where the graph lies below the x-axis, shown in red.

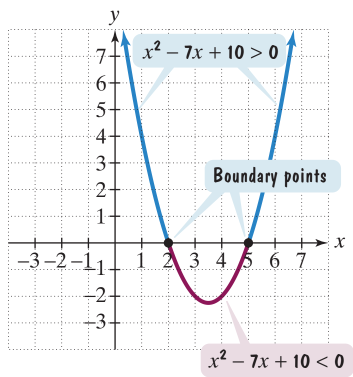


Figure 2.44

Locating the x-intercepts of a polynomial function, f, is an important step in finding the solution set for polynomial inequalities in the form $f(x) < 0$ or $f(x) > 0$. We use the x-intercepts of f as boundary points that divide the real number line into intervals. On each interval, the graph of f is either above the x-axis $[f(x) > 0]$ or below the x-axis $[f(x) < 0]$. For this reason, x-intercepts play a fundamental role in solving polynomial inequalities. The x-intercepts are found by solving the equation $f(x) = 0$.

1 Solve polynomial inequalities.

Procedure for Solving Polynomial Inequalities

1. Express the inequality in the form
$$f(x) < 0 \quad \text{or} \quad f(x) > 0,$$
where f is a polynomial function.
2. Solve the equation $f(x) = 0$. The real solutions are the **boundary points**.
3. Locate these boundary points on a number line, thereby dividing the number line into intervals.
4. Choose one representative number, called a **test value**, within each interval and evaluate f at that number.
   a. If the value of f is positive, then $f(x) > 0$ for all numbers, x, in the interval.
   b. If the value of f is negative, then $f(x) < 0$ for all numbers, x, in the interval.
5. Write the solution set, selecting the interval or intervals that satisfy the given inequality.

This procedure is valid if $<$ is replaced by $\leq$ or $>$ is replaced by $\geq$. However, if the inequality involves $\leq$ or $\geq$, include the boundary points [the solutions of $f(x) = 0$] in the solution set.

EXAMPLE 1 Solving a Polynomial Inequality

Solve and graph the solution set on a real number line: $2x^2 + x > 15$.

Solution

Step 1 Express the inequality in the form $f(x) < 0$ or $f(x) > 0$. We begin by rewriting the inequality so that 0 is on the right side.

$$2x^2 + x > 15 \qquad \text{This is the given inequality.}$$
$$2x^2 + x - 15 > 15 - 15 \qquad \text{Subtract 15 from both sides.}$$
$$2x^2 + x - 15 > 0 \qquad \text{Simplify.}$$

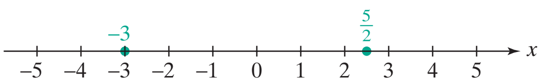
This inequality is equivalent to the one we wish to solve. It is in the form $f(x) > 0$, where $f(x) = 2x^2 + x - 15$.

Step 2 Solve the equation $f(x) = 0$. We find the x-intercepts of $f(x) = 2x^2 + x - 15$ by solving the equation $2x^2 + x - 15 = 0$.

$$2x^2 + x - 15 = 0 \qquad \text{This polynomial equation is a quadratic equation.}$$
$$(2x - 5)(x + 3) = 0 \qquad \text{Factor.}$$
$$2x - 5 = 0 \quad \text{or} \quad x + 3 = 0 \qquad \text{Set each factor equal to 0.}$$
$$x = \tfrac{5}{2} \qquad\qquad x = -3 \qquad \text{Solve for } x.$$

The x-intercepts of f are -3 and $\frac{5}{2}$. We will use these x-intercepts as boundary points on a number line.

Step 3 Locate the boundary points on a number line and separate the line into intervals. The number line with the boundary points is shown as follows:



The boundary points divide the number line into three intervals:

$$(-\infty, -3) \quad \left(-3, \tfrac{5}{2}\right) \quad \left(\tfrac{5}{2}, \infty\right).$$

Step 4 Choose one test value within each interval and evaluate f at that number.

Interval	Test Value	Substitute into $f(x) = 2x^2 + x - 15$	Conclusion
$(-\infty, -3)$	-4	$f(-4) = 2(-4)^2 + (-4) - 15$ $= 13$, positive	$f(x) > 0$ for all x in $(-\infty, -3)$.
$\left(-3, \frac{5}{2}\right)$	0	$f(0) = 2 \cdot 0^2 + 0 - 15$ $= -15$, negative	$f(x) < 0$ for all x in $\left(-3, \frac{5}{2}\right)$.
$\left(\frac{5}{2}, \infty\right)$	3	$f(3) = 2 \cdot 3^2 + 3 - 15$ $= 6$, positive	$f(x) > 0$ for all x in $\left(\frac{5}{2}, \infty\right)$.

Technology

Graphic Connections

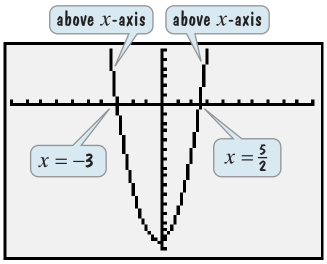
The solution set for

$$2x^2 + x > 15$$

or, equivalently,

$$2x^2 + x - 15 > 0$$

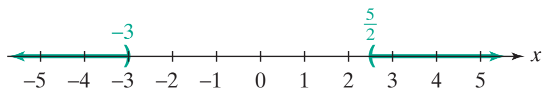
can be verified with a graphing utility. The graph of $f(x) = 2x^2 + x - 15$ was obtained using a $[-10, 10, 1]$ by $[-16, 6, 1]$ viewing rectangle. The graph lies above the x-axis, representing $>$, for all x in $(-\infty, -3)$ or $\left(\frac{5}{2}, \infty\right)$.



Step 5 Write the solution set, selecting the interval or intervals that satisfy the given inequality. We are interested in solving $f(x) > 0$, where $f(x) = 2x^2 + x - 15$. Based on our work in step 4, we see that $f(x) > 0$ for all x in $(-\infty, -3)$ or $\left(\frac{5}{2}, \infty\right)$. Thus, the solution set of the given inequality, $2x^2 + x > 15$, or, equivalently, $2x^2 + x - 15 > 0$, is

$$(-\infty, -3) \cup \left(\tfrac{5}{2}, \infty\right) \text{ or } \left\{x \mid x < -3 \text{ or } x > \tfrac{5}{2}\right\}.$$

The graph of the solution set on a number line is shown as follows:



Check Point 1 Solve and graph the solution set: $x^2 - x > 20$.

EXAMPLE 2 Solving a Polynomial Inequality

Solve and graph the solution set on a real number line: $x^3 + x^2 \le 4x + 4$.

Solution

Step 1 Express the inequality in the form $f(x) \le 0$ or $f(x) \ge 0$. We begin by rewriting the inequality so that 0 is on the right side.

$$x^3 + x^2 \le 4x + 4 \qquad \text{This is the given inequality.}$$

$$x^3 + x^2 - 4x - 4 \le 4x + 4 - 4x - 4 \qquad \text{Subtract } 4x + 4 \text{ from both sides.}$$

$$x^3 + x^2 - 4x - 4 \le 0 \qquad \text{Simplify.}$$

This inequality is equivalent to the one we wish to solve. It is in the form $f(x) \le 0$, where $f(x) = x^3 + x^2 - 4x - 4$.

Step 2 Solve the equation $f(x) = 0$. We find the x-intercepts of $f(x) = x^3 + x^2 - 4x - 4$ by solving the equation $x^3 + x^2 - 4x - 4 = 0$.

$$x^3 + x^2 - 4x - 4 = 0 \qquad \text{This polynomial equation is of degree 3.}$$

$$x^2(x + 1) - 4(x + 1) = 0 \qquad \text{Factor } x^2 \text{ from the first two terms and } -4 \text{ from the last two terms.}$$

$$(x + 1)(x^2 - 4) = 0 \qquad \text{A common factor of } x + 1 \text{ is factored from the expression.}$$

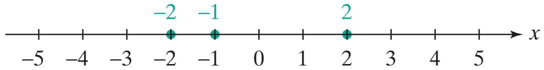
$$(x + 1)(x + 2)(x - 2) = 0 \qquad \text{Factor completely.}$$

$$x + 1 = 0 \quad \text{or} \quad x + 2 = 0 \quad \text{or} \quad x - 2 = 0 \qquad \text{Set each factor equal to 0.}$$

$$x = -1 \qquad x = -2 \qquad x = 2 \qquad \text{Solve for } x.$$

The x-intercepts of f are -2, -1, and 2. We will use these x-intercepts as boundary points on a number line.

Step 3 Locate the boundary points on a number line and separate the line into intervals. The number line with the boundary points is shown as follows:



The boundary points divide the number line into four intervals:

$$(-\infty, -2) \quad (-2, -1) \quad (-1, 2) \quad (2, \infty).$$

Step 4 Choose one test value within each interval and evaluate f at that number.

Interval	Test Value	Substitute into $f(x) = x^3 + x^2 - 4x - 4$	Conclusion
$(-\infty, -2)$	-3	$f(-3) = (-3)^3 + (-3)^2 - 4(-3) - 4$ $= -10$, negative	$f(x) < 0$ for all x in $(-\infty, -2)$.
$(-2, -1)$	-1.5	$f(-1.5) = (-1.5)^3 + (-1.5)^2 - 4(-1.5) - 4$ $= 0.875$, positive	$f(x) > 0$ for all x in $(-2, -1)$.
$(-1, 2)$	0	$f(0) = 0^3 + 0^2 - 4 \cdot 0 - 4$ $= -4$, negative	$f(x) < 0$ for all x in $(-1, 2)$.
$(2, \infty)$	3	$f(3) = 3^3 + 3^2 - 4 \cdot 3 - 4$ $= 20$, positive	$f(x) > 0$ for all x in $(2, \infty)$.

Technology

Graphic Connections

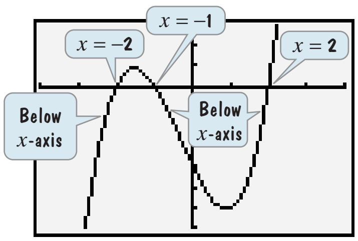
The solution set for

$$x^3 + x^2 \le 4x + 4$$

or, equivalently,

$$x^3 + x^2 - 4x - 4 \le 0$$

can be verified with a graphing utility. The graph of $f(x) = x^3 + x^2 - 4x - 4$ lies on or below the x-axis, representing $\le$, for all x in $(-\infty, -2]$ or $[-1, 2]$.



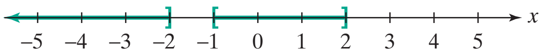
$[-4, 4, 1]$ by $[-7, 3, 1]$

Step 5 Write the solution set, selecting the interval or intervals that satisfy the given inequality. We are interested in solving $f(x) \le 0$, where $f(x) = x^3 + x^2 - 4x - 4$. Based on our work in step 4, we see that $f(x) < 0$ for all x in $(-\infty, -2)$ or $(-1, 2)$. However, because the inequality involves $\le$ (less than or *equal to*), we must also include the solutions of $x^3 + x^2 - 4x - 4 = 0$, namely $-2, -1$, and 2, in the solution set. Thus, the solution set of the given inequality, $x^3 + x^2 \le 4x + 4$, or, equivalently, $x^3 + x^2 - 4x - 4 \le 0$, is

$$(-\infty, -2] \cup [-1, 2]$$

$$\text{or} \quad \{x \mid x \le -2 \text{ or } -1 \le x \le 2\}.$$

The graph of the solution set on a number line is shown as follows:



Check Point 2 Solve and graph the solution set on a real number line: $x^3 + 3x^2 \le x + 3$.

Solving Rational Inequalities

A **rational inequality** is any inequality that can be put into one of the forms

$$f(x) < 0, \quad f(x) > 0, \quad f(x) \le 0, \quad \text{or} \quad f(x) \ge 0,$$

where f is a rational function. An example of a rational inequality is

$$\frac{3x + 3}{2x + 4} > 0.$$

This inequality is in the form $f(x) > 0$, where f is the rational function given by

$$f(x) = \frac{3x + 3}{2x + 4}.$$

The graph of f is shown in **Figure 2.45**.

We can find the x-intercept of f by setting the numerator equal to 0:

$$3x + 3 = 0$$
$$3x = -3$$
$$x = -1.$$

f has an x-intercept at -1 and passes through $(-1, 0)$.

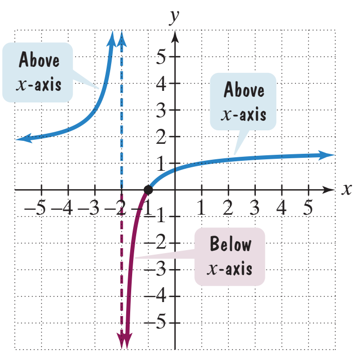


Figure 2.45 The graph of $f(x) = \dfrac{3x + 3}{2x + 4}$

We can determine where f is undefined by setting the denominator equal to 0:

$$\begin{aligned} 2x + 4 &= 0 \\ 2x &= -4 \\ x &= -2. \end{aligned}$$

f is undefined at -2. **Figure 2.45** shows that the function's vertical asymptote is $x = -2$.

By setting both the numerator and the denominator of f equal to 0, we obtained -2 and -1. These numbers separate the x-axis into three intervals: $(-\infty, -2)$, $(-2, -1)$, and $(-1, \infty)$. On each interval, the graph of f is either above the x-axis $[f(x) > 0]$ or below the x-axis $[f(x) < 0]$.

Examine the graph in **Figure 2.45** carefully. Can you see that it is above the x-axis for all x in $(-\infty, -2)$ or $(-1, \infty)$, shown in blue? Thus, the solution set of $\frac{3x + 3}{2x + 4} > 0$ is $(-\infty, -2) \cup (-1, \infty)$. By contrast, the graph of f lies below the x-axis for all x in $(-2, -1)$, shown in red. Thus, the solution set of $\frac{3x + 3}{2x + 4} < 0$ is $(-2, -1)$.

2 Solve rational inequalities.

The first step in solving a rational inequality is to bring all terms to one side, obtaining zero on the other side. Then express the rational function on the nonzero side as a single quotient. The second step is to set the numerator and the denominator of f equal to zero. The solutions of these equations serve as boundary points that separate the real number line into intervals. At this point, the procedure is the same as the one we used for solving polynomial inequalities.

Study Tip

Do not begin solving

$$\frac{x + 1}{x + 3} \geq 2$$

by multiplying both sides by $x + 3$. We do not know if $x + 3$ is positive or negative. Thus, we do not know whether or not to change the sense of the inequality.

EXAMPLE 3 Solving a Rational Inequality

Solve and graph the solution set: $\frac{x + 1}{x + 3} \geq 2$.

Solution

Step 1 Express the inequality so that one side is zero and the other side is a single quotient. We subtract 2 from both sides to obtain zero on the right.

$$\frac{x + 1}{x + 3} \geq 2 \qquad \text{This is the given inequality.}$$

$$\frac{x + 1}{x + 3} - 2 \geq 0 \qquad \text{Subtract 2 from both sides, obtaining 0 on the right.}$$

$$\frac{x + 1}{x + 3} - \frac{2(x + 3)}{x + 3} \geq 0 \qquad \text{The least common denominator is } x + 3. \text{ Express 2 in terms of this denominator.}$$

$$\frac{x + 1 - 2(x + 3)}{x + 3} \geq 0 \qquad \text{Subtract rational expressions.}$$

$$\frac{x + 1 - 2x - 6}{x + 3} \geq 0 \qquad \text{Apply the distributive property.}$$

$$\frac{-x - 5}{x + 3} \geq 0 \qquad \text{Simplify.}$$

This inequality is equivalent to the one we wish to solve. It is in the form $f(x) \geq 0$, where $f(x) = \frac{-x - 5}{x + 3}$.

Step 2 Set the numerator and the denominator of f equal to zero. The real solutions are the boundary points.

$$-x - 5 = 0 \qquad x + 3 = 0 \qquad \text{Set the numerator and denominator equal to 0. These are the values that make the previous quotient zero or undefined.}$$

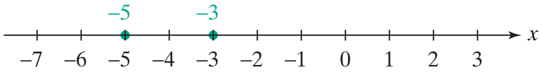
$$x = -5 \qquad x = -3 \qquad \text{Solve for } x.$$

We will use these solutions as boundary points on a number line.

Study Tip

Never include a value that causes a rational function's denominator to equal zero in the solution set of a rational inequality. Division by zero is undefined.

Step 3 Locate the boundary points on a number line and separate the line into intervals. The number line with the boundary points is shown as follows:



The boundary points divide the number line into three intervals:

$$(-\infty, -5) \quad (-5, -3) \quad (-3, \infty).$$

Step 4 Choose one test value within each interval and evaluate *f* at that number.

Interval	Test Value	Substitute into $f(x) = \dfrac{-x-5}{x+3}$	Conclusion
$(-\infty, -5)$	-6	$f(-6) = \dfrac{-(-6)-5}{-6+3} = -\frac{1}{3}$, negative	$f(x) < 0$ for all x in $(-\infty, -5)$.
$(-5, -3)$	-4	$f(-4) = \dfrac{-(-4)-5}{-4+3} = 1$, positive	$f(x) > 0$ for all x in $(-5, -3)$.
$(-3, \infty)$	0	$f(0) = \dfrac{-0-5}{0+3} = -\frac{5}{3}$, negative	$f(x) < 0$ for all x in $(-3, \infty)$.

Technology

Graphic Connections

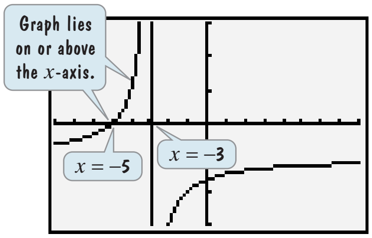
The solution set for

$$\frac{x+1}{x+3} \geq 2$$

or, equivalently,

$$\frac{-x-5}{x+3} \geq 0$$

can be verified with a graphing utility. The graph of $f(x) = \dfrac{-x-5}{x+3}$ lies on or above the x-axis, representing $\geq$, for all x in $[-5, -3)$.



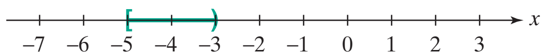
[–8, 8, 1] by [–3, 3, 1]

Step 5 Write the solution set, selecting the interval or intervals that satisfy the given inequality. We are interested in solving $f(x) \geq 0$, where $f(x) = \dfrac{-x-5}{x+3}$. Based on our work in step 4, we see that $f(x) > 0$ for all x in $(-5, -3)$. However, because the inequality involves $\geq$ (greater than or *equal to*), we must also include the solution of $f(x) = 0$, namely the value that we obtained when we set the numerator of f equal to zero. Thus, we must include -5 in the solution set. The solution set of the given inequality is

$$[-5, -3) \text{ or } \{x \mid -5 \leq x < -3\}.$$

−3 causes the denominator of f to equal zero. It must be excluded from the solution set.

The graph of the solution set on a number line is shown as follows:



Check Point 3 Solve and graph the solution set: $\dfrac{2x}{x+1} \geq 1$.

3 Solve problems modeled by polynomial or rational inequalities.

Applications

If you throw an object directly upward, although its path is straight and vertical, its changing height over time can be described by a quadratic function.

The Position Function for a Free-Falling Object Near Earth's Surface

An object that is falling or vertically projected into the air has its height above the ground, $s(t)$, in feet, given by

$$s(t) = -16t^2 + v_0 t + s_0,$$

where v_0 is the original velocity (initial velocity) of the object, in feet per second, t is the time that the object is in motion, in seconds, and s_0 is the original height (initial height) of the object, in feet.

In Example 4, we solve a polynomial inequality in a problem about the position of a free-falling object.

EXAMPLE 4 Using the Position Function

A ball is thrown vertically upward from the top of the Leaning Tower of Pisa (190 feet high) with an initial velocity of 96 feet per second (**Figure 2.46**). During which time period will the ball's height exceed that of the tower?

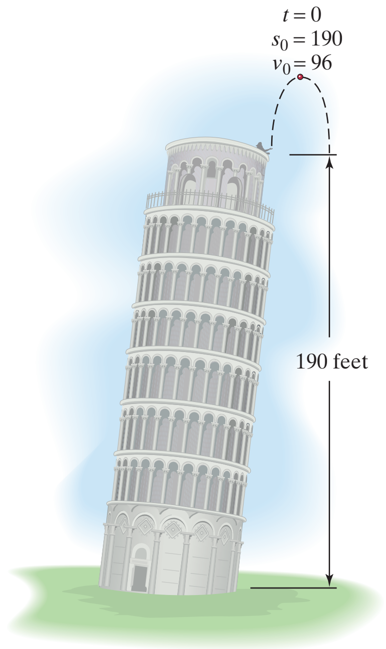


Figure 2.46 Throwing a ball from 190 feet with a velocity of 96 feet per second

Solution

$$s(t) = -16t^2 + v_0 t + s_0$$ This is the position function for a free-falling object.

$$s(t) = -16t^2 + 96t + 190$$ Because v_0 (initial velocity) $= 96$ and s_0 (initial position) $= 190$, substitute these values into the formula.

When will $s(t)$, the ball's height, | exceed that | of the tower?

$$-16t^2 + 96t + 190 > 190$$

$$-16t^2 + 96t + 190 > 190$$ This is the inequality that models the problem's question. We must find t.

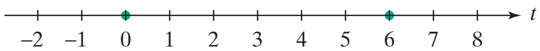
$$-16t^2 + 96t > 0$$ Subtract 190 from both sides. This inequality is in the form $f(t) > 0$, where $f(t) = -16t^2 + 96t$.

$$-16t^2 + 96t = 0$$ Solve the equation $f(t) = 0$.

$$-16t(t - 6) = 0$$ Factor.

$$-16t = 0 \quad \text{or} \quad t - 6 = 0$$ Set each factor equal to 0.

$$t = 0 \qquad\qquad t = 6$$ Solve for t. The boundary points are 0 and 6.



Locate these values on a number line.

The intervals are $(-\infty, 0)$, $(0, 6)$, and $(6, \infty)$. For our purposes, the mathematical model is useful only from $t = 0$ until the ball hits the ground. (By setting $-16t^2 + 96t + 190$ equal to zero, we find $t \approx 7.57$; the ball hits the ground after approximately 7.57 seconds.) Thus, we use $(0, 6)$ and $(6, 7.57)$ for our intervals.

Interval	Test Value	Substitute into $f(t) = -16t^2 + 96t$	Conclusion
$(0, 6)$	1	$f(1) = -16 \cdot 1^2 + 96 \cdot 1$ $= 80$, positive	$f(t) > 0$ for all t in $(0, 6)$.
$(6, 7.57)$	7	$f(7) = -16 \cdot 7^2 + 96 \cdot 7$ $= -112$, negative	$f(t) < 0$ for all t in $(6, 7.57)$.

We are interested in solving $f(t) > 0$, where $f(t) = -16t^2 + 96t$. We see that $f(t) > 0$ for all t in $(0, 6)$. This means that the ball's height exceeds that of the tower between 0 and 6 seconds.

Technology

Graphic Connections

The graphs of

$$y_1 = -16x^2 + 96x + 190$$

and

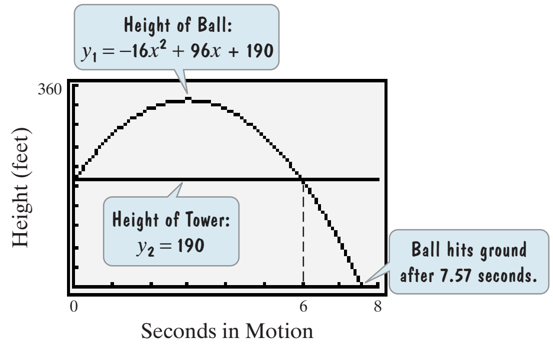
$$y_2 = 190$$

are shown in a

$$[0, 8, 1] \text{ by } [0, 360, 36]$$

seconds in motion

height, in feet



[0, 8, 1] by [0, 360, 36]

viewing rectangle. The graphs show that the ball's height exceeds that of the tower between 0 and 6 seconds.

Check Point 4 An object is propelled straight up from ground level with an initial velocity of 80 feet per second. Its height at time t is modeled by

$$s(t) = -16t^2 + 80t,$$

where the height, $s(t)$, is measured in feet and the time, t, is measured in seconds. In which time interval will the object be more than 64 feet above the ground?

Exercise Set 2.7

Practice Exercises

Solve each polynomial inequality in Exercises 1–42 and graph the solution set on a real number line. Express each solution set in interval notation.

1. $(x - 4)(x + 2) > 0$

2. $(x + 3)(x - 5) > 0$

3. $(x - 7)(x + 3) \le 0$

4. $(x + 1)(x - 7) \le 0$

5. $x^2 - 5x + 4 > 0$

6. $x^2 - 4x + 3 < 0$

7. $x^2 + 5x + 4 > 0$

8. $x^2 + x - 6 > 0$

9. $x^2 - 6x + 9 < 0$

10. $x^2 - 2x + 1 > 0$

11. $3x^2 + 10x - 8 \le 0$

12. $9x^2 + 3x - 2 \ge 0$

13. $2x^2 + x < 15$

14. $6x^2 + x > 1$

15. $4x^2 + 7x < -3$

16. $3x^2 + 16x < -5$

17. $5x \le 2 - 3x^2$

18. $4x^2 + 1 \ge 4x$

19. $x^2 - 4x \ge 0$

20. $x^2 + 2x < 0$

21. $2x^2 + 3x > 0$

22. $3x^2 - 5x \le 0$

23. $-x^2 + x \ge 0$

24. $-x^2 + 2x \ge 0$

25. $x^2 \le 4x - 2$

26. $x^2 \le 2x + 2$

27. $9x^2 - 6x + 1 < 0$

28. $4x^2 - 4x + 1 \ge 0$

29. $(x - 1)(x - 2)(x - 3) \ge 0$

30. $(x + 1)(x + 2)(x + 3) \ge 0$

31. $x(3 - x)(x - 5) \le 0$

32. $x(4 - x)(x - 6) \le 0$

33. $(2 - x)^2\left(x - \frac{7}{2}\right) < 0$

34. $(5 - x)^2\left(x - \frac{13}{2}\right) < 0$

35. $x^3 + 2x^2 - x - 2 \ge 0$

36. $x^3 + 2x^2 - 4x - 8 \ge 0$

37. $x^3 - 3x^2 - 9x + 27 < 0$

38. $x^3 + 7x^2 - x - 7 < 0$

39. $x^3 + x^2 + 4x + 4 > 0$

40. $x^3 - x^2 + 9x - 9 > 0$

41. $x^3 \ge 9x^2$

42. $x^3 \le 4x^2$

Solve each rational inequality in Exercises 43–60 and graph the solution set on a real number line. Express each solution set in interval notation.

43. $\dfrac{x - 4}{x + 3} > 0$

44. $\dfrac{x + 5}{x - 2} > 0$

45. $\dfrac{x + 3}{x + 4} < 0$

46. $\dfrac{x + 5}{x + 2} < 0$

47. $\dfrac{-x + 2}{x - 4} \ge 0$

48. $\dfrac{-x - 3}{x + 2} \le 0$

49. $\dfrac{4 - 2x}{3x + 4} \le 0$

50. $\dfrac{3x + 5}{6 - 2x} \ge 0$

51. $\dfrac{x}{x - 3} > 0$

52. $\dfrac{x + 4}{x} > 0$

53. $\dfrac{(x + 4)(x - 1)}{x + 2} \le 0$

54. $\dfrac{(x + 3)(x - 2)}{x + 1} \le 0$

55. $\dfrac{x + 1}{x + 3} < 2$

56. $\dfrac{x}{x - 1} > 2$

57. $\dfrac{x + 4}{2x - 1} \le 3$

58. $\dfrac{1}{x - 3} < 1$

59. $\dfrac{x - 2}{x + 2} \le 2$

60. $\dfrac{x}{x + 2} \ge 2$

Practice Plus

In Exercises 61–64, find the domain of each function.

61. $f(x) = \sqrt{2x^2 - 5x + 2}$

62. $f(x) = \dfrac{1}{\sqrt{4x^2 - 9x + 2}}$

63. $f(x) = \sqrt{\dfrac{2x}{x + 1} - 1}$

64. $f(x) = \sqrt{\dfrac{x}{2x - 1} - 1}$

Solve each inequality in Exercises 65–70 and graph the solution set on a real number line.

65. $|x^2 + 2x - 36| > 12$

66. $|x^2 + 6x + 1| > 8$

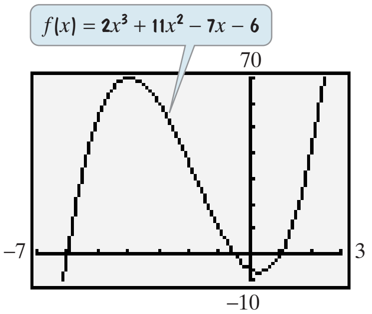
67. $\dfrac{3}{x+3} > \dfrac{3}{x-2}$

68. $\dfrac{1}{x+1} > \dfrac{2}{x-1}$

69. $\dfrac{x^2 - x - 2}{x^2 - 4x + 3} > 0$

70. $\dfrac{x^2 - 3x + 2}{x^2 - 2x - 3} > 0$

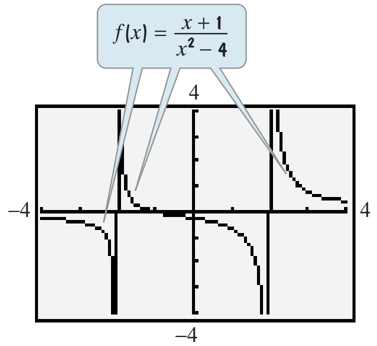
In Exercises 71–72, use the graph of the polynomial function to solve each inequality.



71. $2x^3 + 11x^2 \geq 7x + 6$

72. $2x^3 + 11x^2 < 7x + 6$

In Exercises 73–74, use the graph of the rational function to solve each inequality.



73. $\dfrac{1}{4(x+2)} \leq -\dfrac{3}{4(x-2)}$

74. $\dfrac{1}{4(x+2)} > -\dfrac{3}{4(x-2)}$

Application Exercises

Use the position function

$$s(t) = -16t^2 + v_0 t + s_0$$

(v_0 = initial velocity, s_0 = initial position, t = time)

to answer Exercises 75–76.

75. Divers in Acapulco, Mexico, dive headfirst at 8 feet per second from the top of a cliff 87 feet above the Pacific Ocean. During which time period will the diver's height exceed that of the cliff?

76. You throw a ball straight up from a rooftop 160 feet high with an initial velocity of 48 feet per second. During which time period will the ball's height exceed that of the rooftop?

The functions

$$f(x) = 0.0875x^2 - 0.4x + 66.6$$

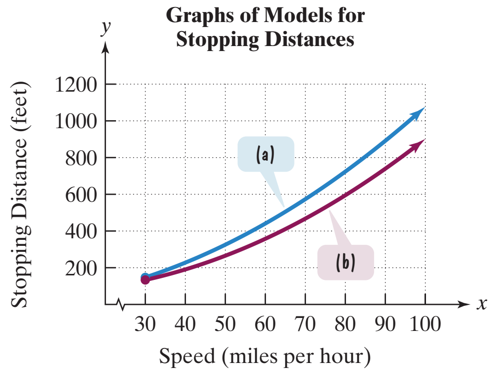
Dry pavement

and

Wet pavement

$$g(x) = 0.0875x^2 + 1.9x + 11.6$$

model a car's stopping distance, $f(x)$ or $g(x)$, in feet, traveling at x miles per hour. Function f models stopping distance on dry pavement and function g models stopping distance on wet pavement. The graphs of these functions are shown for $\{x \mid x \geq 30\}$. Notice that the figure does not specify which graph is the model for dry roads and which is the model for wet roads. Use this information to solve Exercises 77–78.



77. a. Use the given functions to find the stopping distance on dry pavement and the stopping distance on wet pavement for a car traveling at 35 miles per hour. Round to the nearest foot.

b. Based on your answers to part (a), which rectangular coordinate graph shows stopping distances on dry pavement and which shows stopping distances on wet pavement?

c. How well do your answers to part (a) model the actual stopping distances shown in **Figure 2.43** on page 359?

d. Determine speeds on dry pavement requiring stopping distances that exceed the length of one and one-half football fields, or 540 feet. Round to the nearest mile per hour. How is this shown on the appropriate graph of the models?

78. a. Use the given functions to find the stopping distance on dry pavement and the stopping distance on wet pavement for a car traveling at 55 miles per hour. Round to the nearest foot.

b. Based on your answers to part (a), which rectangular coordinate graph shows stopping distances on dry pavement and which shows stopping distances on wet pavement?

c. How well do your answers to part (a) model the actual stopping distances shown in **Figure 2.43** on page 359?

d. Determine speeds on wet pavement requiring stopping distances that exceed the length of one and one-half football fields, or 540 feet. Round to the nearest mile per hour. How is this shown on the appropriate graph of the models?

79. The perimeter of a rectangle is 50 feet. Describe the possible lengths of a side if the area of the rectangle is not to exceed 114 square feet.

80. The perimeter of a rectangle is 180 feet. Describe the possible lengths of a side if the area of the rectangle is not to exceed 800 square feet.

Writing in Mathematics

81. What is a polynomial inequality?

82. What is a rational inequality?

83. If f is a polynomial or rational function, explain how the graph of f can be used to visualize the solution set of the inequality $f(x) < 0$.

Technology Exercises

84. Use a graphing utility to verify your solution sets to any three of the polynomial inequalities that you solved algebraically in Exercises 1–42.

85. Use a graphing utility to verify your solution sets to any three of the rational inequalities that you solved algebraically in Exercises 43–60.

Solve each inequality in Exercises 86–91 using a graphing utility.

86. $x^2 + 3x - 10 > 0$

87. $2x^2 + 5x - 3 \le 0$

88. $x^3 + x^2 - 4x - 4 > 0$

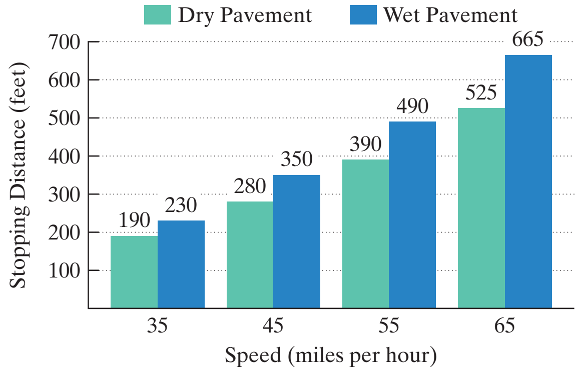
89. $\dfrac{x - 4}{x - 1} \le 0$

90. $\dfrac{x + 2}{x - 3} \le 2$

91. $\dfrac{1}{x + 1} \le \dfrac{2}{x + 4}$

The graph shows stopping distances for trucks at various speeds on dry roads and on wet roads. Use this information to solve Exercises 92–93.

Stopping Distances for Trucks at Selected Speeds



Source: National Highway Traffic Safety Administration

92. a. Use the statistical menu of your graphing utility and the quadratic regression program to obtain the quadratic function that models a truck's stopping distance, $f(x)$, in feet, on dry pavement traveling at x miles per hour. Round the x-coefficient and the constant term to one decimal place.

b. Use the function from part (a) to determine speeds on dry pavement requiring stopping distances that exceed 455 feet. Round to the nearest mile per hour.

93. a. Use the statistical menu of your graphing utility and the quadratic regression program to obtain the quadratic function that models a truck's stopping distance, $f(x)$, in feet, on wet pavement traveling at x miles per hour. Round the x-coefficient and the constant term to one decimal place.

b. Use the function from part (a) to determine speeds on wet pavement requiring stopping distances that exceed 446 feet.

Critical Thinking Exercises

Make Sense? *In Exercises 94–97, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

94. When solving $f(x) > 0$, where f is a polynomial function, I only pay attention to the sign of f at each test value and not the actual function value.

95. I'm solving a polynomial inequality that has a value for which the polynomial function is undefined.

96. Because it takes me longer to come to a stop on a wet road than on a dry road, graph (a) for Exercises 77–78 is the model for stopping distances on wet pavement and graph (b) is the model for stopping distances on dry pavement.

97. I began the solution of the rational inequality $\dfrac{x + 1}{x + 3} \ge 2$ by setting both $x + 1$ and $x + 3$ equal to zero.

In Exercises 98–101, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

98. The solution set of $x^2 > 25$ is $(5, \infty)$.

99. The inequality $\dfrac{x - 2}{x + 3} < 2$ can be solved by multiplying both sides by $x + 3$, resulting in the equivalent inequality $x - 2 < 2(x + 3)$.

100. $(x + 3)(x - 1) \ge 0$ and $\dfrac{x + 3}{x - 1} \ge 0$ have the same solution set.

101. The inequality $\dfrac{x - 2}{x + 3} < 2$ can be solved by multiplying both sides by $(x + 3)^2$, $x \ne -3$, resulting in the equivalent inequality $(x - 2)(x + 3) < 2(x + 3)^2$.

102. Write a polynomial inequality whose solution set is $[-3, 5]$.

103. Write a rational inequality whose solution set is $(-\infty, -4) \cup [3, \infty)$.

In Exercises 104–107, use inspection to describe each inequality's solution set. Do not solve any of the inequalities.

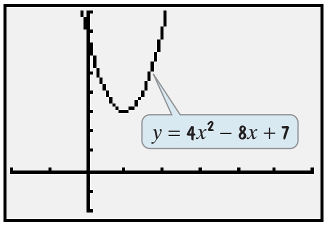
104. $(x - 2)^2 > 0$

105. $(x - 2)^2 \le 0$

106. $(x - 2)^2 < -1$

107. $\dfrac{1}{(x - 2)^2} > 0$

108. The graphing utility screen shows the graph of $y = 4x^2 - 8x + 7$.



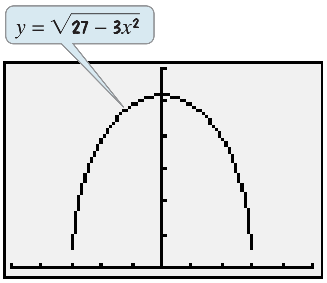
$[-2, 6, 1]$ by $[-2, 8, 1]$

a. Use the graph to describe the solution set of $4x^2 - 8x + 7 > 0$.

b. Use the graph to describe the solution set of $4x^2 - 8x + 7 < 0$.

c. Use an algebraic approach to verify each of your descriptions in parts (a) and (b).

109. The graphing utility screen shows the graph of $y = \sqrt{27 - 3x^2}$. Write and solve a quadratic inequality that explains why the graph only appears for $-3 \le x \le 3$.



$[-5, 5, 1]$ by $[0, 6, 1]$

Preview Exercises

Exercises 110–112 will help you prepare for the material covered in the next section.

110. **a.** If $y = kx^2$, find the value of k using $x = 2$ and $y = 64$.

b. Substitute the value for k into $y = kx^2$ and write the resulting equation.

c. Use the equation from part (b) to find y when $x = 5$.

111. **a.** If $y = \frac{k}{x}$, find the value of k using $x = 8$ and $y = 12$.

b. Substitute the value for k into $y = \frac{k}{x}$ and write the resulting equation.

c. Use the equation from part (b) to find y when $x = 3$.

112. If $S = \frac{kA}{P}$, find the value of k using $A = 60{,}000$, $P = 40$, and $S = 12{,}000$.

Section 2.8 Modeling Using Variation

Objectives

1. Solve direct variation problems.
2. Solve inverse variation problems.
3. Solve combined variation problems.
4. Solve problems involving joint variation.

Have you ever wondered how telecommunication companies estimate the number of phone calls expected per day between two cities? The formula

$$C = \frac{0.02P_1P_2}{d^2}$$

shows that the daily number of phone calls, C, increases as the populations of the cities, P_1 and P_2, in thousands, increase and decreases as the distance, d, between the cities increases.

Certain formulas occur so frequently in applied situations that they are given special names. Variation formulas show how one quantity changes in relation to other quantities. Quantities can vary *directly*, *inversely*, or *jointly*. In this section, we look at situations that can be modeled by each of these kinds of variation. And think of this: The next time you get one of those "all-circuits-are-busy" messages, you will be able to use a variation formula to estimate how many other callers you're competing with for those precious 5-cent minutes.

1 Solve direct variation problems.

Direct Variation

When you swim underwater, the pressure in your ears depends on the depth at which you are swimming. The formula

$$p = 0.43d$$

describes the water pressure, p, in pounds per square inch, at a depth of d feet. We can use this linear function to determine the pressure in your ears at various depths:

If $d = 20$, $p = 0.43(20) = 8.6$. At a depth of 20 feet, water pressure is 8.6 pounds per square inch.

Doubling the depth doubles the pressure.

If $d = 40$, $p = 0.43(40) = 17.2$. At a depth of 40 feet, water pressure is 17.2 pounds per square inch.

Doubling the depth doubles the pressure.

If $d = 80$, $p = 0.43(80) = 34.4$. At a depth of 80 feet, water pressure is 34.4 pounds per square inch.

The formula $p = 0.43d$ illustrates that water pressure is a constant multiple of your underwater depth. If your depth is doubled, the pressure is doubled; if your depth is tripled, the pressure is tripled; and so on. Because of this, the