Rational Functions and Their Graphs

Objectives

1. Find the domains of rational functions.
2. Use arrow notation.
3. Identify vertical asymptotes.
4. Identify horizontal asymptotes.
5. Use transformations to graph rational functions.
6. Graph rational functions.
7. Identify slant asymptotes.
8. Solve applied problems involving rational functions.

Your grandmother appears to be slowing down. Enter … Mega-Grandma! Japanese researchers have developed the robotic exoskeleton shown here to help the elderly and disabled walk and even lift heavy objects like the three 22-pound bags of rice in the photo. It's called the Hybrid Assistive Limb, or HAL. (The inventor has obviously never seen *2001: A Space Odyssey*.) HAL's brain is a computer housed in a back-pack that learns to mimic the wearer's gait and posture. Bioelectric sensors pick up signals transmitted from the brain to the muscles, so it can anticipate movements the moment the wearer thinks of them. A commercial version is available at a hefty cost ranging between $14,000 and $20,000. (*Source*: sanlab.kz.tsukuba.ac.jp)

The cost of manufacturing robotic exoskeletons can be modeled by rational functions. In this section, you will see that high production levels of HAL can eventually make this amazing invention more affordable for the elderly and people with disabilities.

① Find the domains of rational functions.

Rational Functions

Rational functions are quotients of polynomial functions. This means that rational functions can be expressed as

$$f(x) = \frac{p(x)}{q(x)},$$

where p and q are polynomial functions and $q(x) \neq 0$. The **domain** of a rational function is the set of all real numbers except the x-values that make the denominator zero. For example, the domain of the rational function

$$f(x) = \frac{x^2 + 7x + 9}{x(x - 2)(x + 5)}$$

This is $p(x)$.

This is $q(x)$.

is the set of all real numbers except 0, 2, and −5.

EXAMPLE 1 **Finding the Domain of a Rational Function**

Find the domain of each rational function:

a. $f(x) = \dfrac{x^2 - 9}{x - 3}$ **b.** $g(x) = \dfrac{x}{x^2 - 9}$ **c.** $h(x) = \dfrac{x + 3}{x^2 + 9}$.

Solution Rational functions contain division. Because division by 0 is undefined, we must exclude from the domain of each function values of x that cause the polynomial function in the denominator to be 0.

a. The denominator of $f(x) = \dfrac{x^2 - 9}{x - 3}$ is 0 if $x = 3$. Thus, x cannot equal 3.

The domain of f consists of all real numbers except 3. We can express the domain in set-builder or interval notation:

$$\text{Domain of } f = \{x \mid x \neq 3\}$$
$$\text{Domain of } f = (-\infty, 3) \cup (3, \infty).$$

b. The denominator of $g(x) = \dfrac{x}{x^2 - 9}$ is 0 if $x = -3$ or $x = 3$. Thus, the domain of g consists of all real numbers except -3 and 3. We can express the domain in set-builder or interval notation:

$$\text{Domain of } g = \{x \mid x \neq -3, x \neq 3\}$$
$$\text{Domain of } g = (-\infty, -3) \cup (-3, 3) \cup (3, \infty).$$

c. No real numbers cause the denominator of $h(x) = \dfrac{x + 3}{x^2 + 9}$ to equal 0. The domain of h consists of all real numbers.

$$\text{Domain of } h = (-\infty, \infty)$$

⬤

✓ **Check Point | ** Find the domain of each rational function:

a. $f(x) = \dfrac{x^2 - 25}{x - 5}$ **b.** $g(x) = \dfrac{x}{x^2 - 25}$ **c.** $h(x) = \dfrac{x + 5}{x^2 + 25}$.

2 Use arrow notation.

The most basic rational function is the **reciprocal function**, defined by $f(x) = \dfrac{1}{x}$. The denominator of the reciprocal function is zero when $x = 0$, so the domain of f is the set of all real numbers except 0.

Let's look at the behavior of f near the excluded value 0. We start by evaluating $f(x)$ to the left of 0.

x approaches 0 from the left.

x	-1	-0.5	-0.1	-0.01	-0.001
$f(x) = \dfrac{1}{x}$	-1	-2	-10	-100	-1000
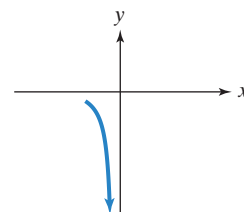
Mathematically, we say that "x approaches 0 from the left." From the table and the accompanying graph, it appears that as x approaches 0 from the left, the function values, $f(x)$, decrease without bound. We say that "$f(x)$ approaches negative infinity." We use a special arrow notation to describe this situation symbolically:
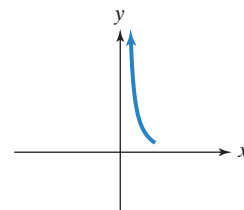
$$\text{As } x \to 0^-, \ f(x) \to -\infty.$$

As x approaches 0 from the left, $f(x)$ approaches negative infinity (that is, the graph falls).

Observe that the minus $(-)$ superscript on the 0 $(x \to 0^-)$ is read "from the left."
Next, we evaluate $f(x)$ to the right of 0.

x approaches 0 from the right.

| x	0.001	0.01	0.1	0.5	1
$f(x) = \dfrac{1}{x}$	1000	100	10	2	1

Mathematically, we say that "x approaches 0 from the right." From the table and the accompanying graph, it appears that as x approaches 0 from the right, the function values, $f(x)$, increase without bound. We say that "$f(x)$ approaches infinity." We again use a special arrow notation to describe this situation symbolically:

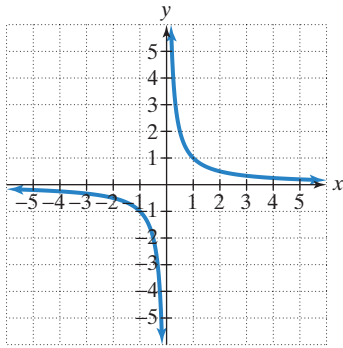$$\text{As } x \to 0^+, \ f(x) \to \infty.$$

As x approaches 0 from the right, $f(x)$ approaches infinity (that is, the graph rises).

Observe that the plus $(+)$ superscript on the 0 $(x \to 0^+)$ is read "from the right."
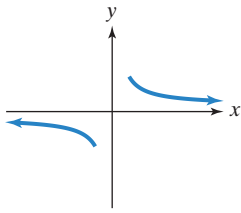
Figure 2.28 $f(x)$ approaches 0 as x increases or decreases without bound.

Figure 2.29 The graph of the reciprocal function $f(x) = \dfrac{1}{x}$

Study Tip

If x is far from 0, then $\dfrac{1}{x}$ is close to 0. By contrast, if x is close to 0, then $\dfrac{1}{x}$ is far from 0.

Now let's see what happens to the function values of $f(x) = \dfrac{1}{x}$ as x gets farther away from the origin. The following tables suggest what happens to $f(x)$ as x increases or decreases without bound.

x increases without bound:				
x	1	10	100	1000
$f(x) = \dfrac{1}{x}$	1	0.1	0.01	0.001

x decreases without bound:				
x	-1	-10	-100	-1000
$f(x) = \dfrac{1}{x}$	-1	-0.1	-0.01	-0.001

It appears that as x increases or decreases without bound, the function values, $f(x)$, are getting progressively closer to 0.

Figure 2.28 illustrates the end behavior of $f(x) = \dfrac{1}{x}$ as x increases or decreases without bound. The graph shows that the function values, $f(x)$, are approaching 0. This means that as x increases or decreases without bound, the graph of f is approaching the horizontal line $y = 0$ (that is, the x-axis). We use arrow notation to describe this situation:

$$\text{As } x \to \infty, \ f(x) \to 0 \qquad \text{and} \qquad \text{as } x \to -\infty, \ f(x) \to 0.$$

> As x approaches infinity (that is, increases without bound), $f(x)$ approaches 0.

> As x approaches negative infinity (that is, decreases without bound), $f(x)$ approaches 0.

Thus, as x approaches infinity $(x \to \infty)$ or as x approaches negative infinity $(x \to -\infty)$, the function values are approaching zero: $f(x) \to 0$.

The graph of the reciprocal function $f(x) = \dfrac{1}{x}$ is shown in **Figure 2.29**. Unlike the graph of a polynomial function, the graph of the reciprocal function has a break and is composed of two distinct branches.

The Reciprocal Function as a Knuckle Tattoo

"I got the tattoo because I like the idea of math not being well behaved. That sounds lame and I really don't mean that in some kind of anarchy-type way. I just think that it's kind of nice that something as perfectly functional as math can kink up around the edges."

Kink up around the edges? On the next page, we'll describe the graphic behavior of the reciprocal function using *asymptotes* rather than kink. Asymptotes are lines that graphs approach but never touch. *Asymptote* comes from the Greek word *asymptotos*, meaning "not meeting."

The arrow notation used throughout our discussion of the reciprocal function is summarized in the following box:

Arrow Notation

Symbol	Meaning
$x \to a^+$	x approaches a from the right.
$x \to a^-$	x approaches a from the left.
$x \to \infty$	x approaches infinity; that is, x increases without bound.
$x \to -\infty$	x approaches negative infinity; that is, x decreases without bound.

In calculus, you will use **limits** to convey ideas involving a function's end behavior or its possible asymptotic behavior. For example, examine the graph of $f(x) = \dfrac{1}{x}$ in **Figure 2.29**, shown on the previous page, and its end behavior to the right. As $x \to \infty$, the values of $f(x)$ approach 0: $f(x) \to 0$. In calculus, this is symbolized by

$$\lim_{x \to \infty} f(x) = 0.$$ This is read "the limit of f(x) as x approaches infinity equals zero."

Another basic rational function is $f(x) = \dfrac{1}{x^2}$. The graph of this even function, with y-axis symmetry and positive function values, is shown in **Figure 2.30**. Like the reciprocal function, the graph has a break and is composed of two distinct branches.

As $x \to 0^-$, $f(x) \to \infty$.
Function values increase without bound.

As $x \to 0^+$, $f(x) \to \infty$.
Function values increase without bound.

As $x \to -\infty$ (decreases without bound),
$f(x) \to 0$.

As $x \to \infty$ (increases without bound),
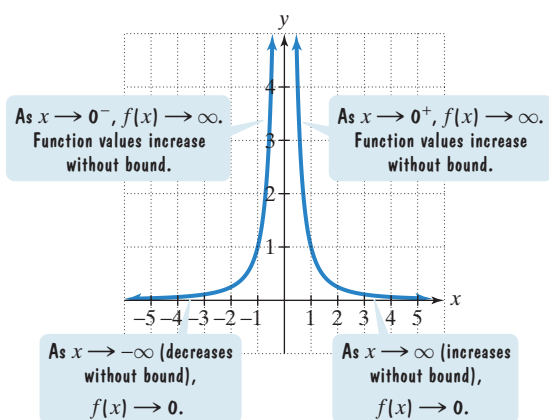$f(x) \to 0$.

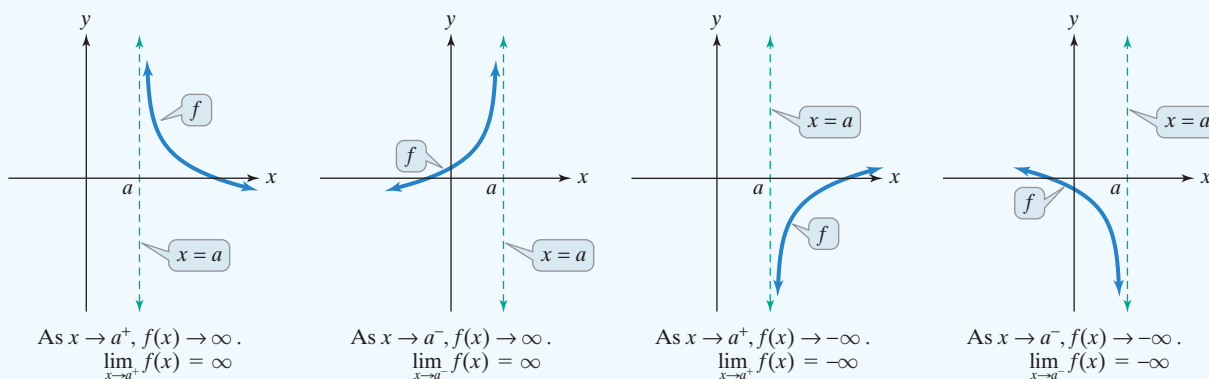**Figure 2.30** The graph of $f(x) = \dfrac{1}{x^2}$

3 Identify vertical asymptotes.

Vertical Asymptotes of Rational Functions

Look again at the graph of $f(x) = \dfrac{1}{x^2}$ in **Figure 2.30**. The curve approaches, but does not touch, the y-axis. The y-axis, or $x = 0$, is said to be a *vertical asymptote* of the graph. A rational function may have no vertical asymptotes, one vertical asymptote, or several vertical asymptotes. The graph of a rational function never intersects a vertical asymptote. We will use dashed lines to show asymptotes.

Definition of a Vertical Asymptote

The line $x = a$ is a **vertical asymptote** of the graph of a function f if $f(x)$ increases or decreases without bound as x approaches a.

As $x \to a^+$, $f(x) \to \infty$.
$\lim\limits_{x \to a^+} f(x) = \infty$

As $x \to a^-$, $f(x) \to \infty$.
$\lim\limits_{x \to a^-} f(x) = \infty$

As $x \to a^+$, $f(x) \to -\infty$.
$\lim\limits_{x \to a^+} f(x) = -\infty$

As $x \to a^-$, $f(x) \to -\infty$.
$\lim\limits_{x \to a^-} f(x) = -\infty$

Thus, as x approaches a from either the left or the right, $f(x) \to \infty$ or $f(x) \to -\infty$.

If the graph of a rational function has vertical asymptotes, they can be located using the following theorem:

Locating Vertical Asymptotes

If $f(x) = \dfrac{p(x)}{q(x)}$ is a rational function in which $p(x)$ and $q(x)$ have no common factors and a is a zero of $q(x)$, the denominator, then $x = a$ is a vertical asymptote of the graph of f.

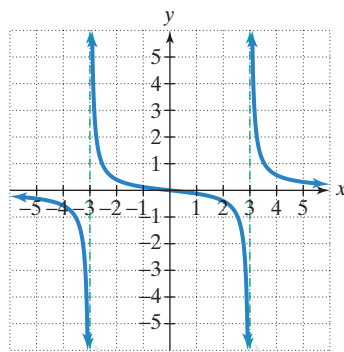**EXAMPLE 2** **Finding the Vertical Asymptotes of a Rational Function**

Find the vertical asymptotes, if any, of the graph of each rational function:

a. $f(x) = \dfrac{x}{x^2 - 9}$
b. $g(x) = \dfrac{x + 3}{x^2 - 9}$
c. $h(x) = \dfrac{x + 3}{x^2 + 9}$.

Vertical asymptote: $x = -3$

Vertical asymptote: $x = 3$

Figure 2.31(a) The graph of $f(x) = \dfrac{x}{x^2 - 9}$ has two vertical asymptotes.

Solution Factoring is usually helpful in identifying zeros of denominators and any common factors in the numerators and denominators.

a. $f(x) = \dfrac{x}{x^2 - 9} = \dfrac{x}{(x + 3)(x - 3)}$

This factor is 0 if $x = -3$.

This factor is 0 if $x = 3$.

There are no common factors in the numerator and the denominator. The zeros of the denominator are -3 and 3. Thus, the lines $x = -3$ and $x = 3$ are the vertical asymptotes for the graph of f. [See **Figure 2.31(a)**.]

b. We will use factoring to see if there are common factors.

$$g(x) = \dfrac{x + 3}{x^2 - 9} = \dfrac{(x + 3)}{(x + 3)(x - 3)} = \dfrac{1}{x - 3}, \text{ provided } x \neq -3$$

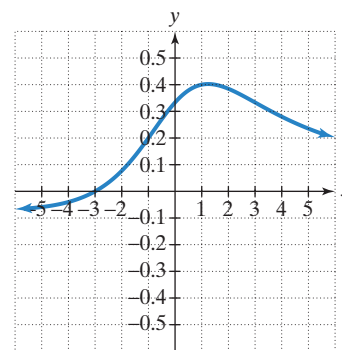There is a common factor, $x + 3$, so simplify.

This denominator is 0 if $x = 3$.

The only zero of the denominator of $g(x)$ in simplified form is 3. Thus, the line $x = 3$ is the only vertical asymptote of the graph of g. [See **Figure 2.31(b)**.]

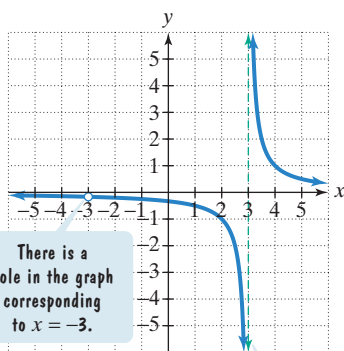**c.** We cannot factor the denominator of $h(x)$ over the real numbers.

$$h(x) = \dfrac{x + 3}{x^2 - 9}$$

No real numbers make this denominator 0.

There is a hole in the graph corresponding to $x = -3$.

Vertical asymptote: $x = 3$

Figure 2.31(b) The graph of $g(x) = \dfrac{x + 3}{x^2 - 9}$ has one vertical asymptote.

The denominator has no real zeros. Thus, the graph of h has no vertical asymptotes. [See **Figure 2.31(c)**.]

Figure 2.31(c) The graph of $h(x) = \dfrac{x + 3}{x^2 + 9}$ has no vertical asymptotes.

✓ Check Point **2** Find the vertical asymptotes, if any, of the graph of each rational function:

a. $f(x) = \dfrac{x}{x^2 - 1}$ **b.** $g(x) = \dfrac{x - 1}{x^2 - 1}$ **c.** $h(x) = \dfrac{x - 1}{x^2 + 1}$.

Technology

The graph of the rational function $f(x) = \dfrac{x}{x^2 - 9}$, drawn by hand in **Figure 2.31(a)**, is graphed below in a $[-5, 5, 1]$ by $[-4, 4, 1]$ viewing rectangle. The graph is shown in connected mode and in dot mode. In connected mode, the graphing utility plots many points and connects the points with curves. In dot mode, the utility plots the same points, but does not connect them.
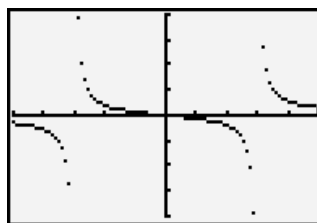
Connected Mode
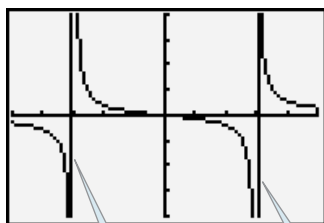
This might appear to be the vertical asymptote $x = -3$, but it is neither vertical nor an asymptote.

This might appear to be the vertical asymptote $x = 3$, but it is neither vertical nor an asymptote.

Dot Mode

The steep lines in connected mode that are "almost" the vertical asymptotes $x = -3$ and $x = 3$ are not part of the graph and do not represent the vertical asymptotes. The graphing utility has incorrectly connected the last point to the left of $x = -3$ with the first point to the right of $x = -3$. It has also incorrectly connected the last point to the left of $x = 3$ with the first point to the right of $x = 3$. The effect is to create two near-vertical segments that look like asymptotes. This erroneous effect does not appear using dot mode.

Study Tip

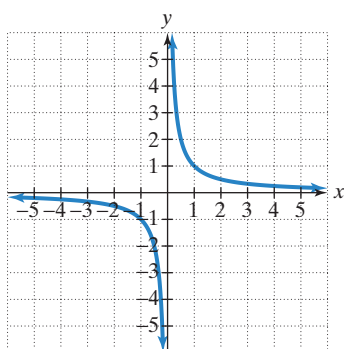It is essential to factor the numerator and the denominator of a rational function to identify possible vertical asymptotes or holes.
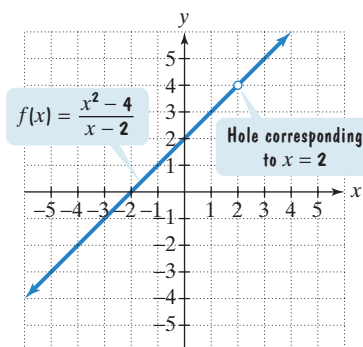
Figure 2.32 A graph with a hole corresponding to the denominator's zero

A value where the denominator of a rational function is zero does not necessarily result in a vertical asymptote. There is a hole corresponding to $x = a$, and not a vertical asymptote, in the graph of a rational function under the following conditions: The value a causes the denominator to be zero, but there is a reduced form of the function's equation in which a does not cause the denominator to be zero.

Consider, for example, the function

$$f(x) = \frac{x^2 - 4}{x - 2}.$$

Because the denominator is zero when $x = 2$, the function's domain is all real numbers except 2. However, there is a reduced form of the equation in which 2 does not cause the denominator to be zero:

$$f(x) = \frac{x^2 - 4}{x - 2} = \frac{(x + 2)(x - 2)}{x - 2} = x + 2, \ x \neq 2.$$

Denominator is zero at $x = 2$.

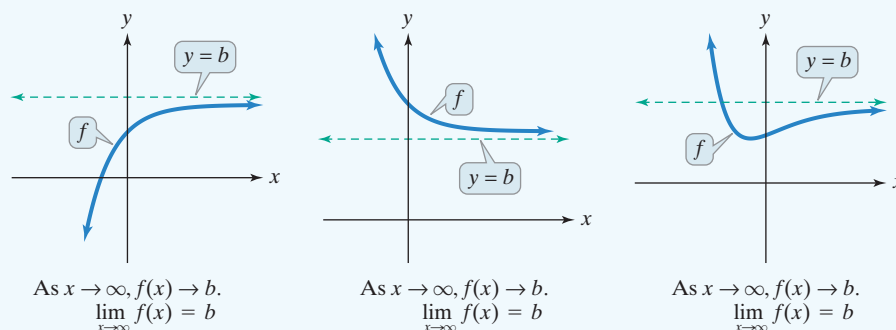In this reduced form, 2 does not result in a zero denominator.

Figure 2.32 shows that the graph has a hole corresponding to $x = 2$. Graphing utilities do not show this feature of the graph.

Horizontal Asymptotes of Rational Functions

④ Identify horizontal asymptotes.

Figure 2.29 The graph of $f(x) = \dfrac{1}{x}$

(repeated)

Figure 2.29, repeated, shows the graph of the reciprocal function $f(x) = \dfrac{1}{x}$. As $x \to \infty$ and as $x \to -\infty$, the function values are approaching 0: $f(x) \to 0$. The line $y = 0$ (that is, the x-axis) is a *horizontal asymptote* of the graph. Many, but not all, rational functions have horizontal asymptotes.

Definition of a Horizontal Asymptote

The line $y = b$ is a **horizontal asymptote** of the graph of a function f if $f(x)$ approaches b as x increases or decreases without bound.

As $x \to \infty, f(x) \to b$.
$\lim\limits_{x \to \infty} f(x) = b$

As $x \to \infty, f(x) \to b$.
$\lim\limits_{x \to \infty} f(x) = b$

As $x \to \infty, f(x) \to b$.
$\lim\limits_{x \to \infty} f(x) = b$

Recall that a rational function may have several vertical asymptotes. By contrast, it can have at most one horizontal asymptote. Although a graph can never intersect a vertical asymptote, it may cross its horizontal asymptote.

If the graph of a rational function has a horizontal asymptote, it can be located using the following theorem:

Study Tip

Unlike identifying possible vertical asymptotes or holes, we do not use factoring to determine a possible horizontal asymptote.

Locating Horizontal Asymptotes

Let f be the rational function given by

$$f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}, \quad a_n \neq 0, b_m \neq 0.$$

The degree of the numerator is n. The degree of the denominator is m.

1. If $n < m$, the x-axis, or $y = 0$, is the horizontal asymptote of the graph of f.

2. If $n = m$, the line $y = \dfrac{a_n}{b_m}$ is the horizontal asymptote of the graph of f.

3. If $n > m$, the graph of f has no horizontal asymptote.
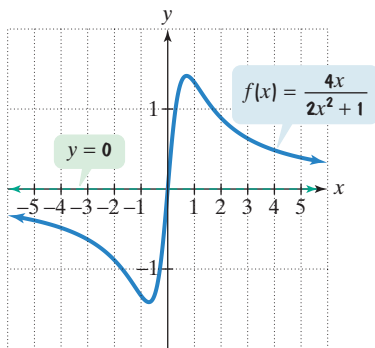
Figure 2.33(a) The horizontal asymptote of the graph is $y = 0$.

Figure 2.33(b) The horizontal asymptote of the graph is $y = 2$.

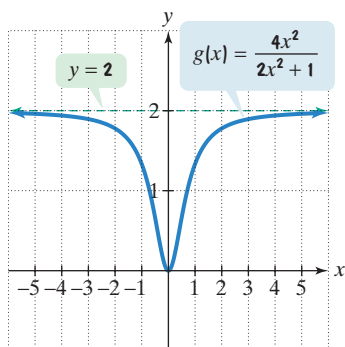**EXAMPLE 3** Finding the Horizontal Asymptote of a Rational Function

Find the horizontal asymptote, if any, of the graph of each rational function:

a. $f(x) = \dfrac{4x}{2x^2 + 1}$ **b.** $g(x) = \dfrac{4x^2}{2x^2 + 1}$ **c.** $h(x) = \dfrac{4x^3}{2x^2 + 1}$.

Solution

a. $f(x) = \dfrac{4x}{2x^2 + 1}$

The degree of the numerator, 1, is less than the degree of the denominator, 2. Thus, the graph of f has the x-axis as a horizontal asymptote. [See **Figure 2.33(a)**.] The equation of the horizontal asymptote is $y = 0$.

b. $g(x) = \dfrac{4x^2}{2x^2 + 1}$

The degree of the numerator, 2, is equal to the degree of the denominator, 2. The leading coefficients of the numerator and denominator, 4 and 2, are used to obtain the equation of the horizontal asymptote. The equation of the horizontal asymptote is $y = \frac{4}{2}$ or $y = 2$. [See **Figure 2.33(b)**.]

c. $h(x) = \dfrac{4x^3}{2x^2 + 1}$

The degree of the numerator, 3, is greater than the degree of the denominator, 2. Thus, the graph of h has no horizontal asymptote. [See **Figure 2.33(c)**.]
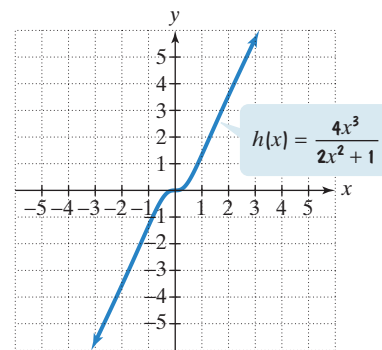
Figure 2.33(c) The graph has no horizontal asymptote.

✓ Check Point **3** Find the horizontal asymptote, if any, of the graph of each rational function:

a. $f(x) = \dfrac{9x^2}{3x^2 + 1}$ **b.** $g(x) = \dfrac{9x}{3x^2 + 1}$ **c.** $h(x) = \dfrac{9x^3}{3x^2 + 1}$.

⑤ Use transformations to graph rational functions.

Using Transformations to Graph Rational Functions

Table 2.2 shows the graphs of two rational functions, $f(x) = \dfrac{1}{x}$ and $f(x) = \dfrac{1}{x^2}$. The dashed green lines indicate the asymptotes.

Table 2.2 Graphs of Common Rational Functions

- Odd function: $f(-x) = -f(x)$
- Origin symmetry
- Even function: $f(-x) = f(x)$
- y-axis symmetry
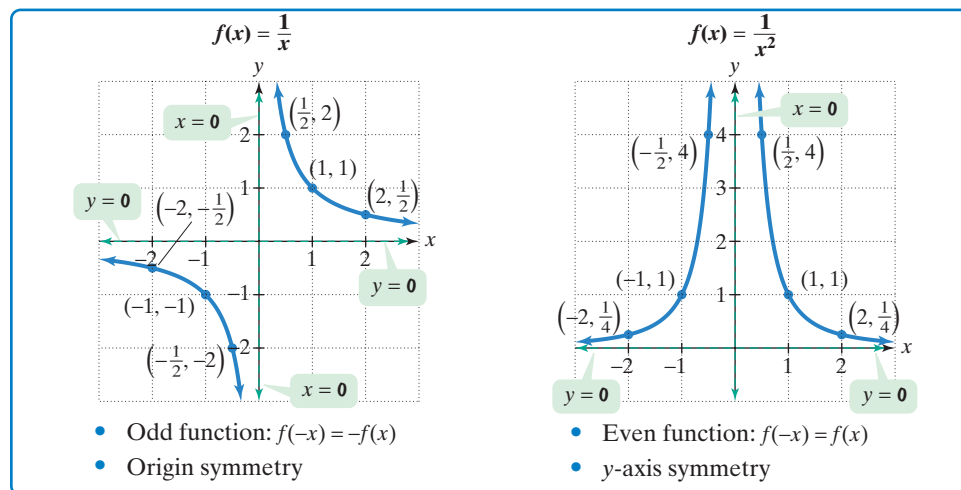
Some rational functions can be graphed using transformations (horizontal shifting, stretching or shrinking, reflecting, vertical shifting) of these two common graphs.

EXAMPLE 4 **Using Transformations to Graph a Rational Function**

Use the graph of $f(x) = \dfrac{1}{x^2}$ to graph $g(x) = \dfrac{1}{(x-2)^2} + 1$.

Solution

Begin with $f(x) = \dfrac{1}{x^2}$.
We've identified two points and the asymptotes.

Graph $y = \dfrac{1}{(x-2)^2}$.
Shift 2 units to the right. Add 2 to each x-coordinate.

The graph of $y = \dfrac{1}{(x-2)^2}$ showing two points and the asymptotes

Graph $g(x) = \dfrac{1}{(x-2)^2} + 1$.
Shift 1 unit up. Add 1 to each y-coordinate.

The graph of $g(x) = \dfrac{1}{(x-2)^2} + 1$ showing two points and the asymptotes
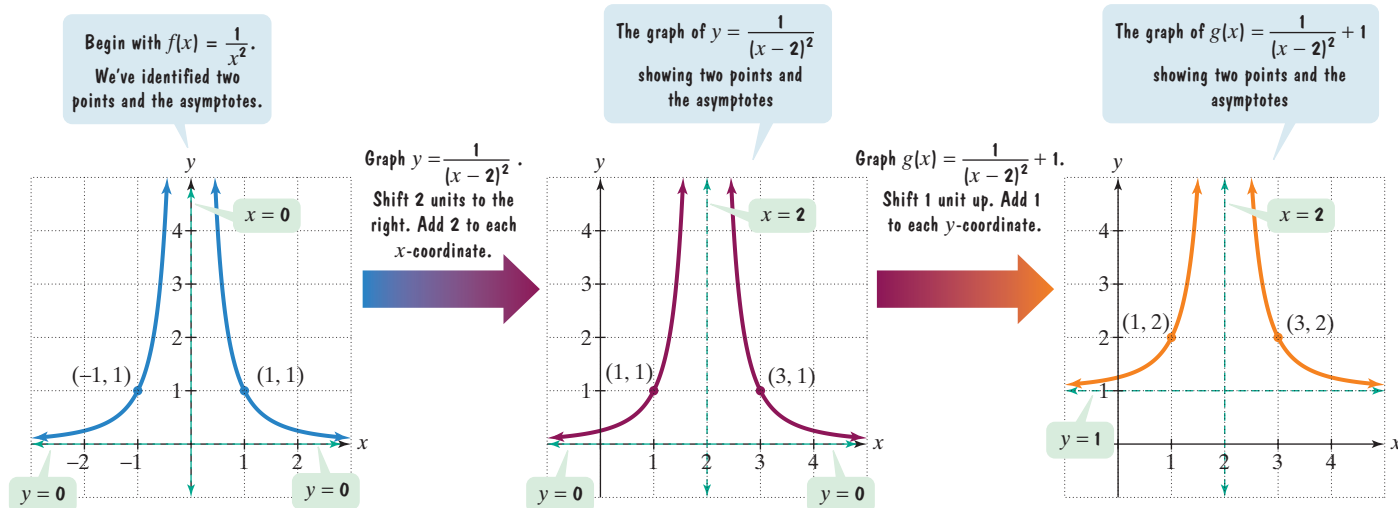
✓ **Check Point 4** Use the graph of $f(x) = \dfrac{1}{x}$ to graph $g(x) = \dfrac{1}{x+2} - 1$.

6 Graph rational functions.

Graphing Rational Functions

Rational functions that are not transformations of $f(x) = \dfrac{1}{x}$ or $f(x) = \dfrac{1}{x^2}$ can be graphed using the following procedure:

Strategy for Graphing a Rational Function

The following strategy can be used to graph

$$f(x) = \frac{p(x)}{q(x)},$$

where p and q are polynomial functions with no common factors.

1. Determine whether the graph of f has symmetry.

$$f(-x) = f(x): \qquad y\text{-axis symmetry}$$
$$f(-x) = -f(x): \qquad \text{origin symmetry}$$

2. Find the y-intercept (if there is one) by evaluating $f(0)$.
3. Find the x-intercepts (if there are any) by solving the equation $p(x) = 0$.
4. Find any vertical asymptote(s) by solving the equation $q(x) = 0$.
5. Find the horizontal asymptote (if there is one) using the rule for determining the horizontal asymptote of a rational function.
6. Plot at least one point between and beyond each x-intercept and vertical asymptote.
7. Use the information obtained previously to graph the function between and beyond the vertical asymptotes.

EXAMPLE 5 **Graphing a Rational Function**

Graph: $f(x) = \dfrac{2x - 1}{x - 1}$.

Solution

Step 1 **Determine symmetry.**

$$f(-x) = \frac{2(-x) - 1}{-x - 1} = \frac{-2x - 1}{-x - 1} = \frac{2x + 1}{x + 1}$$

Because $f(-x)$ does not equal either $f(x)$ or $-f(x)$, the graph has neither y-axis symmetry nor origin symmetry.

Step 2 **Find the y-intercept.** Evaluate $f(0)$.

$$f(0) = \frac{2 \cdot 0 - 1}{0 - 1} = \frac{-1}{-1} = 1$$

The y-intercept is 1, so the graph passes through $(0, 1)$.

Step 3 **Find x-intercept(s).** This is done by solving $p(x) = 0$, where $p(x)$ is the numerator of $f(x)$.

$$2x - 1 = 0 \quad \text{Set the numerator equal to 0.}$$
$$2x = 1 \quad \text{Add 1 to both sides.}$$
$$x = \frac{1}{2} \quad \text{Divide both sides by 2.}$$

The x-intercept is $\frac{1}{2}$, so the graph passes through $\left(\frac{1}{2}, 0\right)$.

Step 4 **Find the vertical asymptote(s).** Solve $q(x) = 0$, where $q(x)$ is the denominator of $f(x)$, thereby finding zeros of the denominator. (Note that the numerator and denominator of $f(x) = \dfrac{2x - 1}{x - 1}$ have no common factors.)

$$x - 1 = 0 \quad \text{Set the denominator equal to 0.}$$
$$x = 1 \quad \text{Add 1 to both sides.}$$

The equation of the vertical asymptote is $x = 1$.

Step 5 **Find the horizontal asymptote.** Because the numerator and denominator of $f(x) = \dfrac{2x - 1}{x - 1}$ have the same degree, 1, the leading coefficients of the numerator and denominator, 2 and 1, respectively, are used to obtain the equation of the horizontal asymptote. The equation is

$$y = \frac{2}{1} = 2.$$

The equation of the horizontal asymptote is $y = 2$.

Step 6 **Plot points between and beyond each x-intercept and vertical asymptote.** With an x-intercept at $\frac{1}{2}$ and a vertical asymptote at $x = 1$, we evaluate the function at $-2, -1, \dfrac{3}{4}, 2,$ and 4.

x	-2	-1	$\dfrac{3}{4}$	2	4
$f(x) = \dfrac{2x - 1}{x - 1}$	$\dfrac{5}{3}$	$\dfrac{3}{2}$	-2	3	$\dfrac{7}{3}$

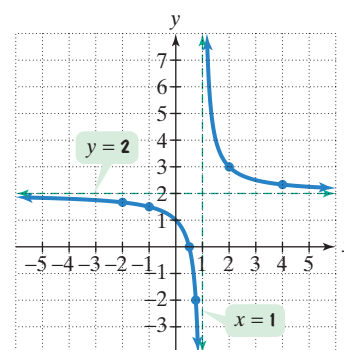**Figure 2.34** shows these points, the y-intercept, the x-intercept, and the asymptotes.

Step 7 **Graph the function.** The graph of $f(x) = \dfrac{2x - 1}{x - 1}$ is shown in **Figure 2.35**.

Technology

The graph of $y = \dfrac{2x - 1}{x - 1}$, obtained using the dot mode in a $[-6, 6, 1]$ by $[-6, 6 \, 1]$ viewing rectangle, verifies that our hand-drawn graph in **Figure 2.35** is correct.
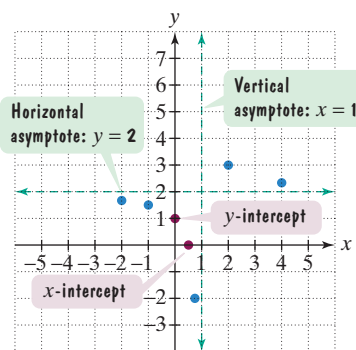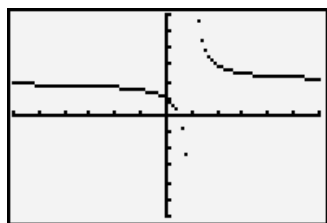
Figure 2.34 Preparing to graph the rational function $f(x) = \dfrac{2x - 1}{x - 1}$

Figure 2.35 The graph of $f(x) = \dfrac{2x - 1}{x - 1}$

✓ **Check Point 5** Graph: $f(x) = \dfrac{3x - 3}{x - 2}$.

EXAMPLE 6 Graphing a Rational Function

Graph: $f(x) = \dfrac{3x^2}{x^2 - 4}$.

Solution

Step 1 Determine symmetry. $f(-x) = \dfrac{3(-x)^2}{(-x)^2 - 4} = \dfrac{3x^2}{x^2 - 4} = f(x)$: The graph of f is symmetric with respect to the y-axis.

Step 2 Find the y-intercept. $f(0) = \dfrac{3 \cdot 0^2}{0^2 - 4} = \dfrac{0}{-4} = 0$: The y-intercept is 0, so the graph passes through the origin.

Step 3 Find the x-intercept(s). $3x^2 = 0$, so $x = 0$: The x-intercept is 0, verifying that the graph passes through the origin.

Step 4 Find the vertical asymptote(s). Set $q(x) = 0$. (Note that the numerator and denominator of $f(x) = \dfrac{3x^2}{x^2 - 4}$ have no common factors.)

$$x^2 - 4 = 0 \qquad \text{Set the denominator equal to 0.}$$
$$x^2 = 4 \qquad \text{Add 4 to both sides.}$$
$$x = \pm 2 \qquad \text{Use the square root property.}$$

The vertical asymptotes are $x = -2$ and $x = 2$.

Step 5 Find the horizontal asymptote. Because the numerator and denominator of $f(x) = \dfrac{3x^2}{x^2 - 4}$ have the same degree, 2, their leading coefficients, 3 and 1, are used to determine the equation of the horizontal asymptote. The equation is $y = \frac{3}{1} = 3$.

Study Tip

Because the graph has y-axis symmetry, it is not necessary to evaluate the even function at -3 and again at 3.

$$f(-3) = f(3) = \tfrac{27}{5}$$

This also applies to evaluation at -1 and 1.

Step 6 Plot points between and beyond each x-intercept and vertical asymptote. With an x-intercept at 0 and vertical asymptotes at $x = -2$ and $x = 2$, we evaluate the function at $-3, -1, 1, 3,$ and 4.

x	-3	-1	1	3	4
$f(x) = \dfrac{3x^2}{x^2 - 4}$	$\dfrac{27}{5}$	-1	-1	$\dfrac{27}{5}$	4

Figure 2.36 shows the points $\left(-3, \dfrac{27}{5}\right), (-1, -1), (1, -1), \left(3, \dfrac{27}{5}\right),$ and $(4, 4),$ the y-intercept, the x-intercept, and the asymptotes.

Step 7   Graph the function. The graph of $f(x) = \dfrac{3x^2}{x^2 - 4}$ is shown in **Figure 2.37**. The y-axis symmetry is now obvious.
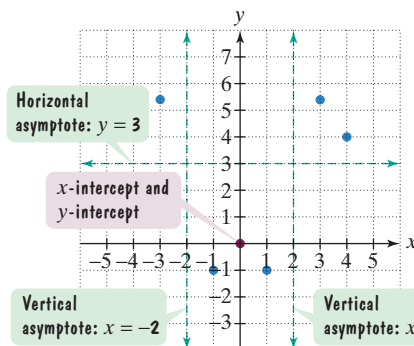
Figure 2.36   Preparing to graph $f(x) = \dfrac{3x^2}{x^2 - 4}$

Figure 2.37   The graph of $f(x) = \dfrac{3x^2}{x^2 - 4}$

✅ **Check Point 6** Graph: $f(x) = \dfrac{2x^2}{x^2 - 9}.$

Example 7 illustrates that not every rational function has vertical and horizontal asymptotes.

EXAMPLE 7 **Graphing a Rational Function**

Graph: $f(x) = \dfrac{x^4}{x^2 + 1}.$

Solution

Step 1   Determine symmetry. $f(-x) = \dfrac{(-x)^4}{(-x)^2 + 1} = \dfrac{x^4}{x^2 + 1} = f(x)$

The graph of f is symmetric with respect to the y-axis.

Step 2   Find the y-intercept. $f(0) = \dfrac{0^4}{0^2 + 1} = \dfrac{0}{1} = 0$: The y-intercept is 0.
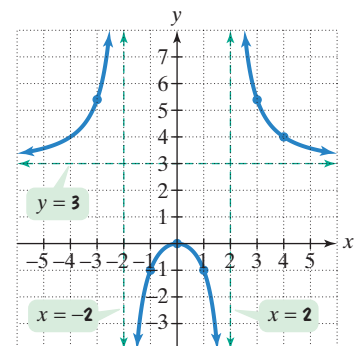
Step 3   Find the x-intercept(s). $x^4 = 0$, so $x = 0$: The x-intercept is 0.

Step 4   Find the vertical asymptote. Set $q(x) = 0$.

$$x^2 + 1 = 0 \qquad \text{Set the denominator equal to 0.}$$
$$x^2 = -1 \qquad \text{Subtract 1 from both sides.}$$

Although this equation has imaginary roots $(x = \pm i)$, there are no real roots. Thus, the graph of f has no vertical asymptotes.

Step 5   Find the horizontal asymptote. Because the degree of the numerator, 4, is greater than the degree of the denominator, 2, there is no horizontal asymptote.

Step 6   Plot points between and beyond each x-intercept and vertical asymptote. With an x-intercept at 0 and no vertical asymptotes, let's look at function values at $-2, -1, 1,$ and 2. You can evaluate the function at 1 and 2. Use y-axis symmetry to obtain function values at -1 and -2:

$$f(-1) = f(1) \text{ and } f(-2) = f(2).$$
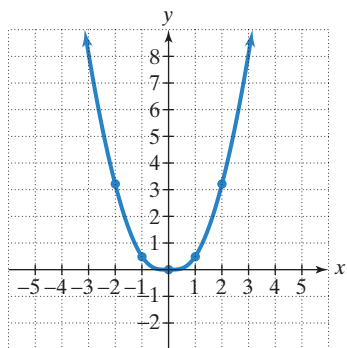
Figure 2.38 The graph of $f(x) = \dfrac{x^4}{x^2 + 1}$

7 Identify slant asymptotes.

x		-2	-1	1	2
$f(x) =$	$\dfrac{x^4}{x^2 + 1}$	$\dfrac{16}{5}$	$\dfrac{1}{2}$	$\dfrac{1}{2}$	$\dfrac{16}{5}$

Step 7 Graph the function. Figure 2.38 shows the graph of f using the points obtained from the table and y-axis symmetry. Notice that as x approaches infinity or negative infinity ($x \to \infty$ or $x \to -\infty$), the function values, $f(x)$, are getting larger without bound $[f(x) \to \infty]$. ●
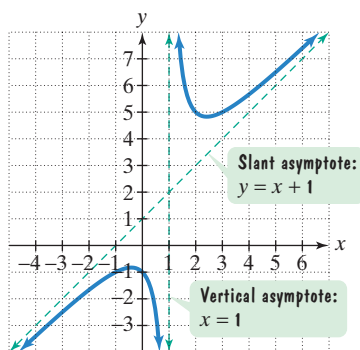
Check Point 7 Graph: $f(x) = \dfrac{x^4}{x^2 + 2}$.

Slant Asymptotes

Examine the graph of

$$f(x) = \frac{x^2 + 1}{x - 1},$$

shown in **Figure 2.39**. Note that the degree of the numerator, 2, is greater than the degree of the denominator, 1. Thus, the graph of this function has no horizontal asymptote. However, the graph has a **slant asymptote,** $y = x + 1$.

The graph of a rational function has a slant asymptote if the degree of the numerator is one more than the degree of the denominator. The equation of the slant asymptote can be found by division. For example, to find the slant asymptote for the graph of $f(x) = \dfrac{x^2 + 1}{x - 1}$, divide $x - 1$ into $x^2 + 1$:

Figure 2.39 The graph of $f(x) = \dfrac{x^2 + 1}{x - 1}$ with a slant asymptote

$$\underline{1\rfloor}\ \ \begin{array}{ccc} 1 & 0 & 1 \\ & 1 & 1 \\ \hline 1 & 1 & 2 \end{array}$$

Remainder

$$\begin{array}{r} 1x + 1 + \dfrac{2}{x-1} \\ x - 1\overline{)x^2 + 0x + 1} \end{array}.$$

Observe that

$$f(x) = \frac{x^2 + 1}{x - 1} = \underbrace{x + 1}_{} + \frac{2}{x - 1}.$$

The equation of the slant asymptote is $y = x + 1.$

As $|x| \to \infty$, the value of $\dfrac{2}{x - 1}$ is approximately 0. Thus, when $|x|$ is large, the function is very close to $y = x + 1 + 0$. This means that as $x \to \infty$ or as $x \to -\infty$, the graph of f gets closer and closer to the line whose equation is $y = x + 1$. The line $y = x + 1$ is a slant asymptote of the graph.

In general, if $f(x) = \dfrac{p(x)}{q(x)}$, p and q have no common factors, and the degree of p is one greater than the degree of q, find the slant asymptote by dividing $q(x)$ into $p(x)$. The division will take the form

$$\frac{p(x)}{q(x)} = mx + b + \frac{\text{remainder}}{q(x)}.$$

Slant asymptote:
$y = mx + b$

The equation of the slant asymptote is obtained by dropping the term with the remainder. Thus, the equation of the slant asymptote is $y = mx + b$.

Slant asymptote:
$y = x - 1$

Vertical asymptote:
$x = 3$
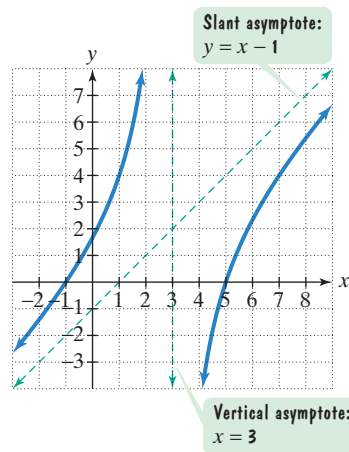
Figure 2.40 The graph of
$f(x) = \dfrac{x^2 - 4x - 5}{x - 3}$

EXAMPLE 8 **Finding the Slant Asymptote of a Rational Function**

Find the slant asymptote of $f(x) = \dfrac{x^2 - 4x - 5}{x - 3}$.

Solution Because the degree of the numerator, 2, is exactly one more than the degree of the denominator, 1, and $x - 3$ is not a factor of $x^2 - 4x - 5$, the graph of f has a slant asymptote. To find the equation of the slant asymptote, divide $x - 3$ into $x^2 - 4x - 5$:

$$\underline{3|} \quad 1 \quad -4 \quad -5$$
$$\qquad \quad \underline{3 \quad -3} \quad \text{Remainder}$$
$$\qquad \quad 1 \quad -1 \quad -8$$

$$1x - 1 - \dfrac{8}{x - 3}$$
$$x - 3\overline{)x^2 - 4x - 5}$$

Drop the remainder term and you'll have the equation of the slant asymptote.

The equation of the slant asymptote is $y = x - 1$. Using our strategy for graphing rational functions, the graph of $f(x) = \dfrac{x^2 - 4x - 5}{x - 3}$ is shown in **Figure 2.40**. ●

⊘ **Check Point 8** Find the slant asymptote of $f(x) = \dfrac{2x^2 - 5x + 7}{x - 2}$.

8 Solve applied problems involving rational functions.

Applications

There are numerous examples of asymptotic behavior in functions that model real-world phenomena. Let's consider an example from the business world. The **cost function**, C, for a business is the sum of its fixed and variable costs:

$$C(x) = (\text{fixed cost}) + cx.$$

Cost per unit times the number of units produced, x

The **average cost** per unit for a company to produce x units is the sum of its fixed and variable costs divided by the number of units produced. The **average cost function** is a rational function that is denoted by $\overline{C}$. Thus,

Cost of producing x units:
fixed plus variable costs

$$\overline{C}(x) = \dfrac{(\text{fixed cost}) + cx}{x}.$$

Number of units produced

EXAMPLE 9 **Average Cost for a Business**

We return to the robotic exoskeleton described in the section opener. Suppose a company that manufactures this invention has a fixed monthly cost of $1,000,000 and that it costs $5000 to produce each robotic system.

  a. Write the cost function, C, of producing x robotic systems.
  b. Write the average cost function, $\overline{C}$, of producing x robotic systems.
  c. Find and interpret $\overline{C}(1000)$, $\overline{C}(10,000)$, and $\overline{C}(100,000)$.
  d. What is the horizontal asymptote for the graph of the average cost function, $\overline{C}$? Describe what this represents for the company.

Solution

  a. The cost function, C, is the sum of the fixed cost and the variable costs.

$$C(x) = 1,000,000 + 5000x$$

Fixed cost is $1,000,000.

Variable cost: $5000 for each robotic system produced

b. The average cost function, $\overline{C}$, is the sum of fixed and variable costs divided by the number of robotic systems produced.

$$\overline{C}(x) = \frac{1,000,000 + 5000x}{x} \quad \text{or} \quad \overline{C}(x) = \frac{5000x + 1,000,000}{x}$$

c. We evaluate $\overline{C}$ at 1000, 10,000, and 100,000, interpreting the results.

$$\overline{C}(1000) = \frac{5000(1000) + 1,000,000}{1000} = 6000$$

The average cost per robotic system of producing 1000 systems per month is $6000.

$$\overline{C}(10,000) = \frac{5000(10,000) + 1,000,000}{10,000} = 5100$$

The average cost per robotic system of producing 10,000 systems per month is $5100.

$$\overline{C}(100,000) = \frac{5000(100,000) + 1,000,000}{100,000} = 5010$$

The average cost per robotic system of producing 100,000 systems per month is $5010. Notice that with higher production levels, the cost of producing each robotic exoskeleton decreases.

d. We developed the average cost function

$$\overline{C}(x) = \frac{5000x + 1,000,000}{x}$$

in which the degree of the numerator, 1, is equal to the degree of the denominator, 1. The leading coefficients of the numerator and denominator, 5000 and 1, are used to obtain the equation of the horizontal asymptote. The equation of the horizontal asymptote is

$$y = \frac{5000}{1} \quad \text{or} \quad y = 5000.$$

The horizontal asymptote is shown in **Figure 2.41**. This means that the more robotic systems produced each month, the closer the average cost per system for the company comes to $5000. The least possible cost per robotic exoskeleton is approaching $5000. Competitively low prices take place with high production levels, posing a major problem for small businesses. ●
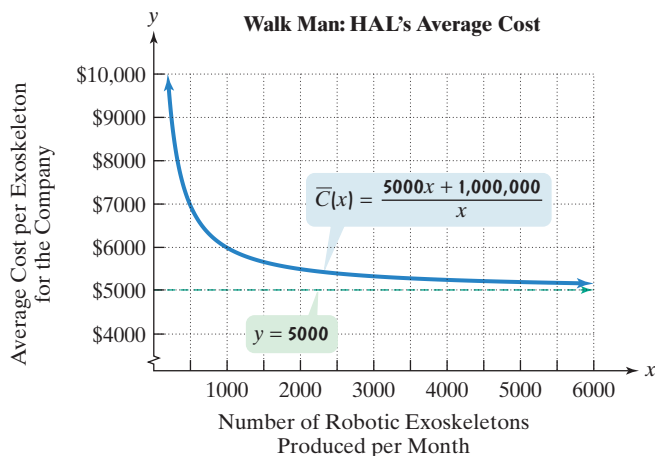

Walk Man: HAL's Average Cost

$\overline{C}(x) = \dfrac{5000x + 1,000,000}{x}$

$y = 5000$

Average Cost per Exoskeleton for the Company

Number of Robotic Exoskeletons Produced per Month

Figure 2.41

✓ Check Point **9** A company is planning to manufacture wheelchairs that are light, fast, and beautiful. The fixed monthly cost will be $500,000 and it will cost $400 to produce each radically innovative chair.

    a. Write the cost function, C, of producing x wheelchairs.

    b. Write the average cost function, $\overline{C}$, of producing x wheelchairs.

    c. Find and interpret $\overline{C}(1000)$, $\overline{C}(10,000)$, and $\overline{C}(100,000)$.

    d. What is the horizontal asymptote for the graph of the average cost function, $\overline{C}$? Describe what this represents for the company.

      If an object moves at an average velocity v, the distance, s, covered in time t is given by the formula

$$s = vt.$$

Thus, *distance = velocity · time*. Objects that move in accordance with this formula are said to be in **uniform motion**. In Example 10, we use a rational function to model time, t, in uniform motion. Solving the uniform motion formula for t, we obtain

$$t = \frac{s}{v}.$$

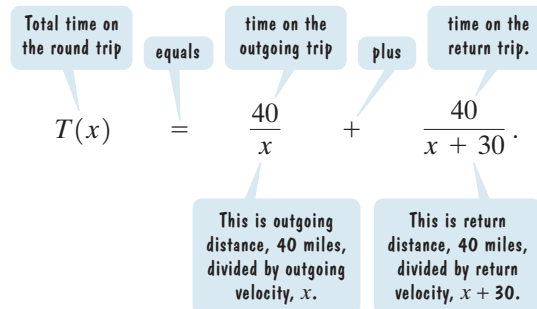Thus, time is the quotient of distance and average velocity.

EXAMPLE 10 **Time Involved in Uniform Motion**

A commuter drove to work a distance of 40 miles and then returned again on the same route. The average velocity on the return trip was 30 miles per hour faster than the average velocity on the outgoing trip. Express the total time required to complete the round trip, T, as a function of the average velocity on the outgoing trip, x.

Solution As specified, the average velocity on the outgoing trip is represented by x. Because the average velocity on the return trip was 30 miles per hour faster than the average velocity on the outgoing trip, let

$$x + 30 = \text{the average velocity on the return trip.}$$

The sentence that we use as a verbal model to write our rational function is

Total time on the round trip	equals	time on the outgoing trip	plus	time on the return trip.
$T(x)$	$=$	$\dfrac{40}{x}$	$+$	$\dfrac{40}{x + 30}$.

This is outgoing distance, 40 miles, divided by outgoing velocity, x.

This is return distance, 40 miles, divided by return velocity, $x + 30$.

The function that expresses the total time required to complete the round trip is

$$T(x) = \frac{40}{x} + \frac{40}{x + 30}.$$

Once you have modeled a problem's conditions with a function, you can use a graphing utility to explore the function's behavior. For example, let's graph the function in Example 10. Because it seems unlikely that an average outgoing velocity exceeds 60 miles per hour with an average return velocity that is 30 miles per hour faster, we graph the function for $0 \le x \le 60$. **Figure 2.42** shows the graph of $T(x) = \dfrac{40}{x} + \dfrac{40}{x + 30}$ in a $[0, 60, 3]$ by $[0, 10, 1]$ viewing rectangle. Notice that the function is decreasing on the interval $(0,60)$. This shows decreasing times with increasing average velocities. Can you see that $x = 0$, or the y-axis, is a vertical asymptote? This indicates that close to an outgoing average velocity of zero miles per hour, the round trip will take nearly forever: $\lim_{x \to 0^+} T(x) = \infty$.
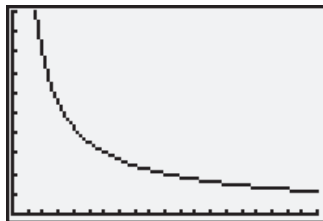
Figure 2.42 The graph of $T(x) = \dfrac{40}{x} + \dfrac{40}{x + 30}$. As average velocity increases, time for the trip decreases: $\lim_{x \to \infty} T(x) = 0$.

✓ Check Point **10** A commuter drove to work a distance of 20 miles and then returned again on the same route. The average velocity on the return trip was 10 miles per hour slower than the average velocity on the outgoing trip. Express the total time required to complete the round trip, T, as a function of the average velocity on the outgoing trip, x.

Exercise Set 2.6

Practice Exercises

In Exercises 1–8, find the domain of each rational function.

1. $f(x) = \dfrac{5x}{x - 4}$

2. $f(x) = \dfrac{7x}{x - 8}$

3. $g(x) = \dfrac{3x^2}{(x - 5)(x + 4)}$

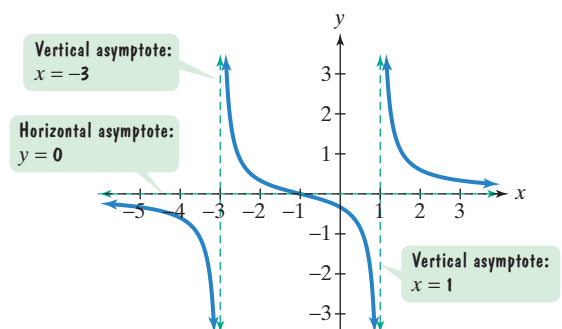**4.** $g(x) = \dfrac{2x^2}{(x - 2)(x + 6)}$

5. $h(x) = \dfrac{x + 7}{x^2 - 49}$

**6.** $h(x) = \dfrac{x + 8}{x^2 - 64}$

7. $f(x) = \dfrac{x + 7}{x^2 + 49}$

8. $f(x) = \dfrac{x + 8}{x^2 + 64}$

Use the graph of the rational function in the figure shown to complete each statement in Exercises 9–14.

Vertical asymptote: $x = -3$

Horizontal asymptote: $y = 0$

Vertical asymptote: $x = 1$

9. As $x \to -3^-$, $f(x) \to$ _____.

10. As $x \to -3^+$, $f(x) \to$ _____.

11. As $x \to 1^-$, $f(x) \to$ _____.

12. As $x \to 1^+$, $f(x) \to$ _____.

13. As $x \to -\infty$, $f(x) \to$ _____.

14. As $x \to \infty$, $f(x) \to$ _____.

Use the graph of the rational function in the figure shown to complete each statement in Exercises 15–20.
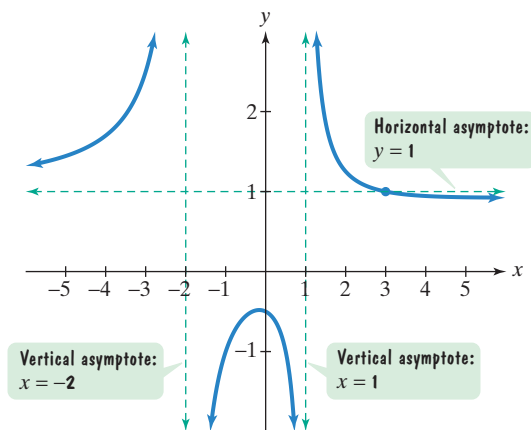
15. As $x \to 1^+$, $f(x) \to$ _____.

16. As $x \to 1^-$, $f(x) \to$ _____.

17. As $x \to -2^+$, $f(x) \to$ _____.

18. As $x \to -2^-$, $f(x) \to$ _____.

19. As $x \to \infty$, $f(x) \to$ _____.

20. As $x \to -\infty$, $f(x) \to$ _____.

In Exercises 21–28, find the vertical asymptotes, if any, of the graph of each rational function.

21. $f(x) = \dfrac{x}{x+4}$

22. $f(x) = \dfrac{x}{x-3}$

23. $g(x) = \dfrac{x+3}{x(x+4)}$

24. $g(x) = \dfrac{x+3}{x(x-3)}$

25. $h(x) = \dfrac{x}{x(x+4)}$

26. $h(x) = \dfrac{x}{x(x-3)}$

27. $r(x) = \dfrac{x}{x^2+4}$

28. $r(x) = \dfrac{x}{x^2+3}$

In Exercises 29–36, find the horizontal asymptote, if any, of the graph of each rational function.

29. $f(x) = \dfrac{12x}{3x^2+1}$

30. $f(x) = \dfrac{15x}{3x^2+1}$

31. $g(x) = \dfrac{12x^2}{3x^2+1}$

32. $g(x) = \dfrac{15x^2}{3x^2+1}$

33. $h(x) = \dfrac{12x^3}{3x^2+1}$

34. $h(x) = \dfrac{15x^3}{3x^2+1}$

35. $f(x) = \dfrac{-2x+1}{3x+5}$

36. $f(x) = \dfrac{-3x+7}{5x-2}$

In Exercises 37–48, use transformations of $f(x) = \dfrac{1}{x}$ or $f(x) = \dfrac{1}{x^2}$ to graph each rational function.

37. $g(x) = \dfrac{1}{x-1}$

38. $g(x) = \dfrac{1}{x-2}$

39. $h(x) = \dfrac{1}{x}+2$

40. $h(x) = \dfrac{1}{x}+1$

41. $g(x) = \dfrac{1}{x+1}-2$

42. $g(x) = \dfrac{1}{x+2}-2$

43. $g(x) = \dfrac{1}{(x+2)^2}$

44. $g(x) = \dfrac{1}{(x+1)^2}$

45. $h(x) = \dfrac{1}{x^2}-4$

46. $h(x) = \dfrac{1}{x^2}-3$

47. $h(x) = \dfrac{1}{(x-3)^2}+1$

48. $h(x) = \dfrac{1}{(x-3)^2}+2$

In Exercises 49–70, follow the seven steps on page 347 to graph each rational function.

49. $f(x) = \dfrac{4x}{x-2}$

50. $f(x) = \dfrac{3x}{x-1}$

51. $f(x) = \dfrac{2x}{x^2-4}$

52. $f(x) = \dfrac{4x}{x^2-1}$

53. $f(x) = \dfrac{2x^2}{x^2-1}$

54. $f(x) = \dfrac{4x^2}{x^2-9}$

55. $f(x) = \dfrac{-x}{x+1}$

56. $f(x) = \dfrac{-3x}{x+2}$

57. $f(x) = -\dfrac{1}{x^2-4}$

58. $f(x) = -\dfrac{2}{x^2-1}$

59. $f(x) = \dfrac{2}{x^2+x-2}$

60. $f(x) = \dfrac{-2}{x^2-x-2}$

61. $f(x) = \dfrac{2x^2}{x^2+4}$

62. $f(x) = \dfrac{4x^2}{x^2+1}$

63. $f(x) = \dfrac{x+2}{x^2+x-6}$

64. $f(x) = \dfrac{x-4}{x^2-x-6}$

65. $f(x) = \dfrac{x^4}{x^2+2}$

66. $f(x) = \dfrac{2x^4}{x^2+1}$

67. $f(x) = \dfrac{x^2+x-12}{x^2-4}$

68. $f(x) = \dfrac{x^2}{x^2+x-6}$

69. $f(x) = \dfrac{3x^2+x-4}{2x^2-5x}$

70. $f(x) = \dfrac{x^2-4x+3}{(x+1)^2}$

*In Exercises 71–78, **a.** Find the slant asymptote of the graph of each rational function and **b.** Follow the seven-step strategy and use the slant asymptote to graph each rational function.*

71. $f(x) = \dfrac{x^2-1}{x}$

72. $f(x) = \dfrac{x^2-4}{x}$

73. $f(x) = \dfrac{x^2+1}{x}$

74. $f(x) = \dfrac{x^2+4}{x}$

75. $f(x) = \dfrac{x^2+x-6}{x-3}$

76. $f(x) = \dfrac{x^2-x+1}{x-1}$

77. $f(x) = \dfrac{x^3+1}{x^2+2x}$

78. $f(x) = \dfrac{x^3-1}{x^2-9}$

Practice Plus

In Exercises 79–84, the equation for f is given by the simplified expression that results after performing the indicated operation. Write the equation for f and then graph the function.

79. $\dfrac{5x^2}{x^2-4} \cdot \dfrac{x^2+4x+4}{10x^3}$

80. $\dfrac{x-5}{10x-2} \div \dfrac{x^2-10x+25}{25x^2-1}$

81. $\dfrac{x}{2x+6} - \dfrac{9}{x^2-9}$

82. $\dfrac{2}{x^2+3x+2} - \dfrac{4}{x^2+4x+3}$

83. $\dfrac{1-\dfrac{3}{x+2}}{1+\dfrac{1}{x-2}}$

84. $\dfrac{x-\dfrac{1}{x}}{x+\dfrac{1}{x}}$

In Exercises 85–88, use long division to rewrite the equation for g in the form

$$\text{quotient} + \frac{\text{remainder}}{\text{divisor}}.$$

Then use this form of the function's equation and transformations of $f(x) = \dfrac{1}{x}$ to graph g.

85. $g(x) = \dfrac{2x + 7}{x + 3}$

86. $g(x) = \dfrac{3x + 7}{x + 2}$

87. $g(x) = \dfrac{3x - 7}{x - 2}$

88. $g(x) = \dfrac{2x - 9}{x - 4}$

Application Exercises

89. A company is planning to manufacture mountain bikes. The fixed monthly cost will be $100,000 and it will cost $100 to produce each bicycle.

   a. Write the cost function, C, of producing x mountain bikes.

   b. Write the average cost function, $\overline{C}$, of producing x mountain bikes.

   c. Find and interpret $\overline{C}(500)$, $\overline{C}(1000)$, $\overline{C}(2000)$, and $\overline{C}(4000)$.

   d. What is the horizontal asymptote for the graph of the average cost function, $\overline{C}$? Describe what this means in practical terms.

90. A company that manufactures running shoes has a fixed monthly cost of $300,000. It costs $30 to produce each pair of shoes.

   a. Write the cost function, C, of producing x pairs of shoes.

   b. Write the average cost function, $\overline{C}$, of producing x pairs of shoes.

   c. Find and interpret $\overline{C}(1000)$, $\overline{C}(10,000)$, and $\overline{C}(100,000)$.

   d. What is the horizontal asymptote for the graph of the average cost function, $\overline{C}$? Describe what this represents for the company.

91. The function

$$f(x) = \frac{6.5x^2 - 20.4x + 234}{x^2 + 36}$$

models the pH level, $f(x)$, of the human mouth x minutes after a person eats food containing sugar. The graph of this function is shown in the figure.

$$f(x) = \frac{6.5x^2 - 20.4x + 234}{x^2 + 36}$$

Number of Minutes after Eating Food Containing Sugar
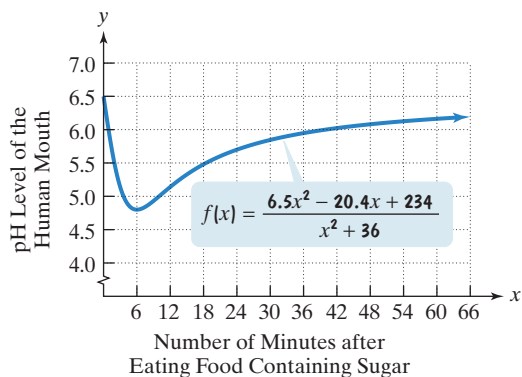
   a. Use the graph to obtain a reasonable estimate, to the nearest tenth, of the pH level of the human mouth 42 minutes after a person eats food containing sugar.

   b. After eating sugar, when is the pH level the lowest? Use the function's equation to determine the pH level, to the nearest tenth, at this time.

   c. According to the graph, what is the normal pH level of the human mouth?

   d. What is the equation of the horizontal asymptote associated with this function? Describe what this means in terms of the mouth's pH level over time.

   e. Use the graph to describe what happens to the pH level during the first hour.

92. A drug is injected into a patient and the concentration of the drug in the bloodstream is monitored. The drug's concentration, $C(t)$, in milligrams per liter, after t hours is modeled by

$$C(t) = \frac{5t}{t^2 + 1}.$$

The graph of this rational function, obtained with a graphing utility, is shown in the figure.

$$y = \frac{5x}{x^2 + 1}$$

[0, 10, 1] by [0, 3, 1]
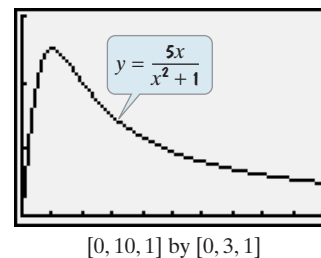
   a. Use the graph to obtain a reasonable estimate of the drug's concentration after 3 hours.

   b. Use the function's equation displayed in the voice balloon to determine the drug's concentration after 3 hours.

   c. Use the function's equation to find the horizontal asymptote for the graph. Describe what this means about the drug's concentration in the patient's bloodstream as time increases.

*Among all deaths from a particular disease, the percentage that are smoking related (21–39 cigarettes per day) is a function of the disease's **incidence ratio**. The incidence ratio describes the number of times more likely smokers are than nonsmokers to die from the disease. The following table shows the incidence ratios for heart disease and lung cancer for two age groups.*
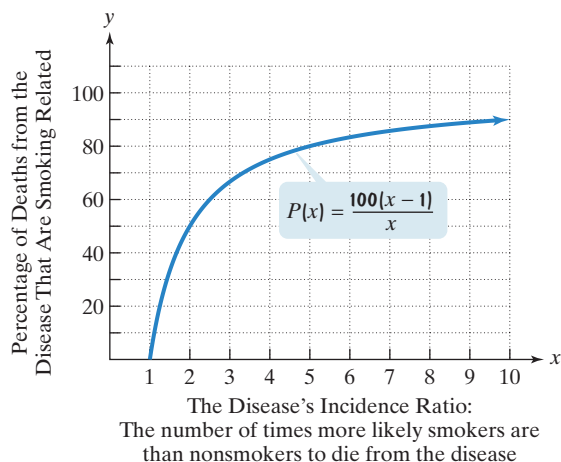
Incidence Ratios		
	Heart Disease	**Lung Cancer**
Ages 55–64	1.9	10
Ages 65–74	1.7	9

Source: Alexander M. Walker, *Observations and Inference,* Epidemiology Resources Inc., 1991.

For example, the incidence ratio of 9 in the table means that smokers between the ages of 65 and 74 are 9 times more likely than nonsmokers in the same age group to die from lung cancer. The rational function

$$P(x) = \frac{100(x - 1)}{x}$$

models the percentage of smoking-related deaths among all deaths from a disease, $P(x)$, in terms of the disease's incidence ratio, x. The graph of the rational function is shown at the top of the next page. Use this function to solve Exercises 93–96.

$$P(x) = \frac{100(x-1)}{x}$$

The Disease's Incidence Ratio:
The number of times more likely smokers are
than nonsmokers to die from the disease

93. Find $P(10)$. Describe what this means in terms of the incidence ratio, 10, given in the table. Identify your solution as a point on the graph.

94. Find $P(9)$. Round to the nearest percent. Describe what this means in terms of the incidence ratio, 9, given in the table. Identify your solution as a point on the graph.

95. What is the horizontal asymptote of the graph? Describe what this means about the percentage of deaths caused by smoking with increasing incidence ratios.

96. According to the model and its graph, is there a disease for which all deaths are caused by smoking? Explain your answer.

97. The bar graph shows the amount, in billions of dollars, that the United States government spent on human resources and total budget outlays for six selected years. (Human resources include education, health, Medicare, Social Security, and veterans benefits and services.)

Federal Budget Expenditures on Human Resources
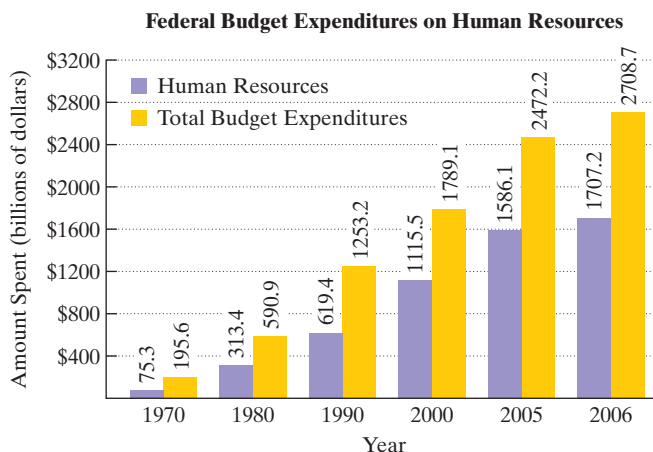

Source: Office of Management and Budget

The function $p(x) = 11x^2 + 40x + 1040$ models the amount, $p(x)$, in billions of dollars, that the United States government spent on human resources x years after 1970. The function $q(x) = 12x^2 + 230x + 2190$ models total budget expenditures, $q(x)$, in billions of dollars, x years after 1970.

a. Use p and q to write a rational function that models the fraction of total budget outlays spent on human resources x years after 1970.

b. Use the data displayed by the bar graph to find the percentage of federal expenditures spent on human resources in 2006. Round to the nearest percent.

c. Use the rational function from part (a) to find the percentage of federal expenditures spent on human resources in 2006. Round to the nearest percent. Does this underestimate or overestimate the actual percent that you found in part (b)? By how much?

d. What is the equation of the horizontal asymptote associated with the rational function in part (a)? If trends modeled by the function continue, what percentage of the federal budget will be spent on human resources over time? Round to the nearest percent. Does this projection seem realistic? Why or why not?

Exercises 98–101 involve writing a rational function that models a problem's conditions.

98. You drive from your home to a vacation resort 600 miles away. You return on the same highway. The average velocity on the return trip is 10 miles per hour slower than the average velocity on the outgoing trip. Express the total time required to complete the round trip, T, as a function of the average velocity on the outgoing trip, x.

99. A tourist drives 90 miles along a scenic highway and then takes a 5-mile walk along a hiking trail. The average velocity driving is nine times that while hiking. Express the total time for driving and hiking, T, as a function of the average velocity on the hike, x.

100. A contractor is constructing the house shown in the figure. The cross section up to the roof is in the shape of a rectangle. The area of the rectangular floor of the house is 2500 square feet. Express the perimeter of the rectangular floor, P, as a function of the width of the rectangle, x.
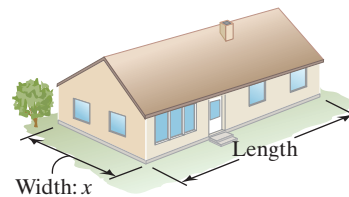
101. The figure shows a page with 1-inch margins at the top and the bottom and half-inch side margins. A publishing company is willing to vary the page dimensions subject to the condition that the printed area of the page is 50 square inches. Express the total area of the page, A, as a function of the width of the rectangle containing the print, x.
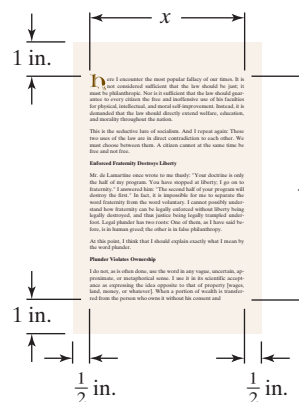
Writing in Mathematics

102. What is a rational function?

103. Use everyday language to describe the graph of a rational function f such that as $x \to -\infty$, $f(x) \to 3$.

104. Use everyday language to describe the behavior of a graph near its vertical asymptote if $f(x) \to \infty$ as $x \to -2^-$ and $f(x) \to -\infty$ as $x \to -2^+$.

105. If you are given the equation of a rational function, explain how to find the vertical asymptotes, if any, of the function's graph.

106. If you are given the equation of a rational function, explain how to find the horizontal asymptote, if any, of the function's graph.

107. Describe how to graph a rational function.

108. If you are given the equation of a rational function, how can you tell if the graph has a slant asymptote? If it does, how do you find its equation?

109. Is every rational function a polynomial function? Why or why not? Does a true statement result if the two adjectives *rational* and *polynomial* are reversed? Explain.

110. Although your friend has a family history of heart disease, he smokes, on average, 25 cigarettes per day. He sees the table showing incidence ratios for heart disease (see Exercises 93–96) and feels comfortable that they are less than 2, compared to 9 and 10 for lung cancer. He claims that all family deaths have been from heart disease and decides not to give up smoking. Use the given function and its graph to describe some additional information not given in the table that might influence his decision.

Technology Exercises

111. Use a graphing utility to verify any five of your hand-drawn graphs in Exercises 37–78.

112. Use a graphing utility to graph $y = \dfrac{1}{x}$, $y = \dfrac{1}{x^3}$, and $\dfrac{1}{x^5}$ in the same viewing rectangle. For odd values of n, how does changing n affect the graph of $y = \dfrac{1}{x^n}$?

113. Use a graphing utility to graph $y = \dfrac{1}{x^2}$, $y = \dfrac{1}{x^4}$, and $y = \dfrac{1}{x^6}$ in the same viewing rectangle. For even values of n, how does changing n affect the graph of $y = \dfrac{1}{x^n}$?

114. Use a graphing utility to graph

$$f(x) = \frac{x^2 - 4x + 3}{x - 2} \quad \text{and} \quad g(x) = \frac{x^2 - 5x + 6}{x - 2}.$$

What differences do you observe between the graph of f and the graph of g? How do you account for these differences?

115. The rational function

$$f(x) = \frac{27{,}725(x - 14)}{x^2 + 9} - 5x$$

models the number of arrests, $f(x)$, per 100,000 drivers, for driving under the influence of alcohol, as a function of a driver's age, x.

   a. Graph the function in a $[0, 70, 5]$ by $[0, 400, 20]$ viewing rectangle.

   b. Describe the trend shown by the graph.

   c. Use the ZOOM and TRACE features or the maximum function feature of your graphing utility to find the age that corresponds to the greatest number of arrests. How many arrests, per 100,000 drivers, are there for this age group?

Critical Thinking Exercises

Make Sense? *In Exercises 116–119, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

116. I've graphed a rational function that has two vertical asymptotes and two horizontal asymptotes.

117. My graph of $y = \dfrac{x - 1}{(x - 1)(x - 2)}$ has vertical asymptotes at $x = 1$ and $x = 2$.

118. The function

$$f(x) = \frac{1.96x + 3.14}{3.04x + 21.79}$$

models the fraction of nonviolent prisoners in New York State prisons x years after 1980. I can conclude from this equation that over time the percentage of nonviolent prisoners will exceed 60%.

119. As production level increases, the average cost for a company to produce each unit of its product also increases.

In Exercises 120–123, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

120. The graph of a rational function cannot have both a vertical asymptote and a horizontal asymptote.

121. It is possible to have a rational function whose graph has no y-intercept.

122. The graph of a rational function can have three vertical asymptotes.

123. The graph of a rational function can never cross a vertical asymptote.

In Exercises 124–127, write the equation of a rational function $f(x) = \dfrac{p(x)}{q(x)}$ *having the indicated properties, in which the degrees of p and q are as small as possible. More than one correct function may be possible. Graph your function using a graphing utility to verify that it has the required properties.*

124. f has a vertical asymptote given by $x = 3$, a horizontal asymptote $y = 0$, y-intercept at -1, and no x-intercept.

125. f has vertical asymptotes given by $x = -2$ and $x = 2$, a horizontal asymptote $y = 2$, y-intercept at $\frac{9}{2}$, x-intercepts at -3 and 3, and y-axis symmetry.

126. f has a vertical asymptote given by $x = 1$, a slant asymptote whose equation is $y = x$, y-intercept at 2, and x-intercepts at -1 and 2.

127. f has no vertical, horizontal, or slant asymptotes, and no x-intercepts.

Preview Exercises

Exercises 128–130 will help you prepare for the material covered in the next section.

128. Solve: $2x^2 + x = 15$.

129. Solve: $x^3 + x^2 = 4x + 4$.

130. Simplify: $\dfrac{x + 1}{x + 3} - 2$.