Technology Exercise

65. For each equation that you solved in Exercises 43–46, use a graphing utility to graph the polynomial function defined by the left side of the equation. Use end behavior to obtain a complete graph. Then use the graph's x-intercepts to verify your solutions.

Critical Thinking Exercises

Make Sense? *In Exercises 66–69, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

66. When performing the division $(x^5 + 1) \div (x + 1)$, there's no need for me to follow all the steps involved in polynomial long division because I can work the problem in my head and see that the quotient must be $x^4 + 1$.

67. Every time I divide polynomials using synthetic division, I am using a highly condensed form of the long division procedure where omitting the variables and exponents does not involve the loss of any essential data.

68. The only nongraphic method that I have for evaluating a function at a given value is to substitute that value into the function's equation.

69. I found the zeros of function f, but I still need to find the solutions of the equation $f(x) = 0$.

In Exercises 70–73, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

70. If a trinomial in x of degree 6 is divided by a trinomial in x of degree 3, the degree of the quotient is 2.

71. Synthetic division can be used to find the quotient of $10x^3 - 6x^2 + 4x - 1$ and $x - \frac{1}{2}$.

72. Any problem that can be done by synthetic division can also be done by the method for long division of polynomials.

73. If a polynomial long-division problem results in a remainder that is a whole number, then the divisor is a factor of the dividend.

74. Find k so that $4x + 3$ is a factor of

$$20x^3 + 23x^2 - 10x + k.$$

75. When $2x^2 - 7x + 9$ is divided by a polynomial, the quotient is $2x - 3$ and the remainder is 3. Find the polynomial.

76. Find the quotient of $x^{3n} + 1$ and $x^n + 1$.

77. Synthetic division is a process for dividing a polynomial by $x - c$. The coefficient of x in the divisor is 1. How might synthetic division be used if you are dividing by $2x - 4$?

78. Use synthetic division to show that 5 is a solution of

$$x^4 - 4x^3 - 9x^2 + 16x + 20 = 0.$$

Then solve the polynomial equation.

Preview Exercises

Exercises 79–81 will help you prepare for the material covered in the next section.

79. Solve: $x^2 + 4x - 1 = 0$.

80. Solve: $x^2 + 4x + 6 = 0$.

81. Let $f(x) = a_n(x^4 - 3x^2 - 4)$. If $f(3) = -150$, determine the value of a_n.

Section 2.5 Zeros of Polynomial Functions

Objectives

1. Use the Rational Zero Theorem to find possible rational zeros.
2. Find zeros of a polynomial function.
3. Solve polynomial equations.
4. Use the Linear Factorization Theorem to find polynomials with given zeros.
5. Use Descartes's Rule of Signs.

You stole my formula!

Tartaglia's Secret Formula for One Solution of $x^3 + mx = n$

$$x = \sqrt[3]{\sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3} + \frac{n}{2}} - \sqrt[3]{\sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3} - \frac{n}{2}}$$

Popularizers of mathematics are sharing bizarre stories that are giving math a secure place in popular culture. One episode, able to compete with the wildest fare served up by television talk shows and the tabloids, involves three Italian mathematicians and, of all things, zeros of polynomial functions.

Tartaglia (1499–1557), poor and starving, has found a formula that gives a root for a third-degree polynomial equation. Cardano (1501–1576) begs Tartaglia to reveal the secret formula, wheedling it from him with the promise he will find the impoverished Tartaglia a patron. Then Cardano publishes his famous work *Ars Magna*, in which he presents Tartaglia's formula as his own. Cardano uses his most talented student, Ferrari (1522–1565), who derived a formula for a root of a fourth-degree polynomial equation, to falsely accuse Tartaglia of plagiarism. The dispute becomes violent and Tartaglia is fortunate to escape alive.

The noise from this "You Stole My Formula" episode is quieted by the work of French mathematician Evariste Galois (1811–1832). Galois proved that there is no general formula for finding roots of polynomial equations of degree 5 or higher. There are, however, methods for finding roots. In this section, we study methods for finding zeros of polynomial functions. We begin with a theorem that plays an important role in this process.

Study Tip

Be sure you are familiar with the various kinds of zeros of polynomial functions. Here's a quick example:

$$f(x) = (x+3)(2x-1)(x+\sqrt{2})(x-\sqrt{2})(x-4+5i)(x-4-5i).$$

Zeros: $-3,\quad \frac{1}{2},\quad -\sqrt{2},\quad \sqrt{2},\quad 4-5i,\quad 4+5i$

Rational zeros (-3, $\frac{1}{2}$); Irrational zeros ($-\sqrt{2}$, $\sqrt{2}$); Complex imaginary zeros ($4-5i$, $4+5i$)

Real zeros (-3, $\frac{1}{2}$, $-\sqrt{2}$, $\sqrt{2}$); Nonreal zeros ($4-5i$, $4+5i$)

1 Use the Rational Zero Theorem to find possible rational zeros.

The Rational Zero Theorem

The Rational Zero Theorem provides us with a tool that we can use to make a list of all possible rational zeros of a polynomial function. Equivalently, the theorem gives all possible rational roots of a polynomial equation. Not every number in the list will be a zero of the function, but every rational zero of the polynomial function will appear somewhere in the list.

The Rational Zero Theorem

If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has *integer* coefficients and $\frac{p}{q}$ (where $\frac{p}{q}$ is reduced to lowest terms) is a rational zero of f, then p is a factor of the constant term, a_0, and q is a factor of the leading coefficient, a_n.

You can explore the "why" behind the Rational Zero Theorem in Exercise 92 of Exercise Set 2.5. For now, let's see if we can figure out what the theorem tells us about possible rational zeros. To use the theorem, list all the integers that are factors of the constant term, a_0. Then list all the integers that are factors of the leading coefficient, a_n. Finally, list all possible rational zeros:

$$\text{Possible rational zeros} = \frac{\text{Factors of the constant term}}{\text{Factors of the leading coefficient}}.$$

EXAMPLE 1 Using the Rational Zero Theorem

List all possible rational zeros of $f(x) = -x^4 + 3x^2 + 4$.

Solution The constant term is 4. We list all of its factors: $\pm1, \pm2, \pm4$. The leading coefficient is -1. Its factors are ±1.

Factors of the constant term, 4: $\pm1, \quad \pm2, \quad \pm4$

Factors of the leading coefficient, -1: ±1

Because

$$\text{Possible rational zeros} = \frac{\text{Factors of the constant term}}{\text{Factors of the leading coefficient}},$$

we must take each number in the first row, $\pm1, \pm2, \pm4$, and divide by each number in the second row, ±1.

$$\text{Possible rational zeros} = \frac{\text{Factors of } 4}{\text{Factors of } -1} = \frac{\pm1, \pm2, \pm4}{\pm1} = \pm1, \quad \pm2, \quad \pm4$$

Divide ±1 by ±1. Divide ±2 by ±1. Divide ±4 by ±1.

There are six possible rational zeros. The graph of $f(x) = -x^4 + 3x^2 + 4$ is shown in **Figure 2.26**. The x-intercepts are -2 and 2. Thus, -2 and 2 are the actual rational zeros.

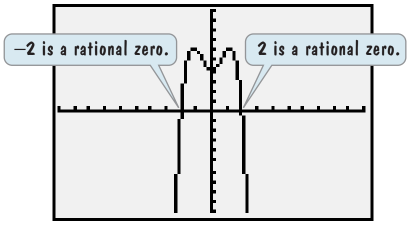


Figure 2.26 The graph of $f(x) = -x^4 + 3x^2 + 4$ shows that -2 and 2 are rational zeros.

Study Tip

Always keep in mind the relationship among zeros, roots, and x-intercepts. The zeros of a function f are the roots, or solutions, of the equation $f(x) = 0$. Furthermore, the real zeros, or real roots, are the x-intercepts of the graph of f.

Check Point 1 List all possible rational zeros of

$$f(x) = x^3 + 2x^2 - 5x - 6.$$

EXAMPLE 2 Using the Rational Zero Theorem

List all possible rational zeros of $f(x) = 15x^3 + 14x^2 - 3x - 2$.

Solution The constant term is -2 and the leading coefficient is 15.

$$\text{Possible rational zeros} = \frac{\text{Factors of the constant term, } -2}{\text{Factors of the leading coefficient, } 15} = \frac{\pm1, \pm2}{\pm1, \pm3, \pm5, \pm15}$$

$$= \pm1, \quad \pm2, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{1}{15}, \pm\frac{2}{15}$$

Divide ±1 and ±2 by ±1. Divide ±1 and ±2 by ±3. Divide ±1 and ±2 by ±5. Divide ±1 and ±2 by ±15.

There are 16 possible rational zeros. The actual solution set of

$$15x^3 + 14x^2 - 3x - 2 = 0$$

is $\left\{-1, -\frac{1}{3}, \frac{2}{5}\right\}$, which contains three of the 16 possible zeros.

Check Point 2 List all possible rational zeros of

$$f(x) = 4x^5 + 12x^4 - x - 3.$$

2 Find zeros of a polynomial function.

How do we determine which (if any) of the possible rational zeros are rational zeros of the polynomial function? To find the first rational zero, we can use a trial-and-error process involving synthetic division: If $f(x)$ is divided by $x - c$ and the remainder is zero, then c is a zero of f. After we identify the first rational zero, we use the result of the synthetic division to factor the original polynomial. Then we set each factor equal to zero to identify any additional rational zeros.

EXAMPLE 3 Finding Zeros of a Polynomial Function

Find all zeros of $f(x) = x^3 + 2x^2 - 5x - 6$.

Solution We begin by listing all possible rational zeros.
Possible rational zeros

$$= \frac{\text{Factors of the constant term, } -6}{\text{Factors of the leading coefficient, } 1} = \frac{\pm1, \pm2, \pm3, \pm6}{\pm1} = \pm1, \pm2, \pm3, \pm6$$

Divide the eight numbers in the numerator by ±1.

Now we will use synthetic division to see if we can find a rational zero among the possible rational zeros $\pm1, \pm2, \pm3, \pm6$. Keep in mind that if $f(x)$ is divided by $x - c$ and the remainder is zero, then c is a zero of f. Let's start by testing 1. If 1 is not a rational zero, then we will test other possible rational zeros.

Test 1.

Coefficients of $f(x) = x^3 + 2x^2 - 5x - 6$

Possible rational zero

1	1	2	−5	−6
		1	3	−2
	1	3	−2	−8

The nonzero remainder shows that 1 is not a zero.

Test 2.

Coefficients of $f(x) = x^3 + 2x^2 - 5x - 6$

Possible rational zero

2	1	2	−5	−6
		2	8	6
	1	4	3	0

The zero remainder shows that 2 is a zero.

The zero remainder tells us that 2 is a zero of the polynomial function $f(x) = x^3 + 2x^2 - 5x - 6$. Equivalently, 2 is a solution, or root, of the polynomial equation $x^3 + 2x^2 - 5x - 6 = 0$. Thus, $x - 2$ is a factor of the polynomial. The first three numbers in the bottom row of the synthetic division on the right, 1, 4, and 3, give the coefficients of the other factor. This factor is $x^2 + 4x + 3$.

$$x^3 + 2x^2 - 5x - 6 = 0$$
Finding the zeros of $f(x) = x^3 + 2x^2 - 5x - 6$ is the same as finding the roots of this equation.

$$(x - 2)(x^2 + 4x + 3) = 0$$
Factor using the result from the synthetic division.

$$(x - 2)(x + 3)(x + 1) = 0$$
Factor completely.

$$x - 2 = 0 \quad \text{or} \quad x + 3 = 0 \quad \text{or} \quad x + 1 = 0$$
Set each factor equal to zero.

$$x = 2 \qquad x = -3 \qquad x = -1$$
Solve for x.

The solution set is $\{-3, -1, 2\}$. The zeros of f are -3, -1, and 2.

Check Point 3 Find all zeros of

$$f(x) = x^3 + 8x^2 + 11x - 20.$$

Our work in Example 3 involved finding zeros of a third-degree polynomial function. The Rational Zero Theorem is a tool that allows us to rewrite such functions as products of two factors, one linear and one quadratic. Zeros of the quadratic factor are found by factoring, the quadratic formula, or the square root property.

EXAMPLE 4 Finding Zeros of a Polynomial Function

Find all zeros of $f(x) = x^3 + 7x^2 + 11x - 3$.

Solution We begin by listing all possible rational zeros.

$$\text{Possible rational zeros} = \frac{\text{Factors of the constant term, } -3}{\text{Factors of the leading coefficient, } 1} = \frac{\pm1, \pm3}{\pm1} = \pm1, \pm3$$

Now we will use synthetic division to see if we can find a rational zero among the four possible rational zeros.

Test 1.

1	1	7	11	−3
		1	8	19
	1	8	19	16

Test −1.

−1	1	7	11	−3
		−1	−6	−5
	1	6	5	−8

Test 3.

3	1	7	11	−3
		3	30	123
	1	10	41	120

Test −3.

−3	1	7	11	−3
		−3	−12	3
	1	4	−1	0

Test −3. (repeated)

−3	1	7	11	−3
		−3	−12	3
	1	4	−1	0

The zero remainder when testing −3, repeated in the margin, tells us that −3 is a zero of the polynomial function $f(x) = x^3 + 7x^2 + 11x - 3$. To find all zeros of f, we proceed as follows:

$$x^3 + 7x^2 + 11x - 3 = 0$$

Finding the zeros of f is the same thing as finding the roots of $f(x) = 0$.

$$(x + 3)(x^2 + 4x - 1) = 0$$

This result is from the last synthetic division, repeated in the margin. The first three numbers in the bottom row, 1, 4, and −1, give the coefficients of the second factor.

$$x + 3 = 0 \quad \text{or} \quad x^2 + 4x - 1 = 0$$

Set each factor equal to 0.

$$x = -3.$$

Solve the linear equation.

We can use the quadratic formula to solve $x^2 + 4x - 1 = 0$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We use the quadratic formula because $x^2 + 4x - 1$ cannot be factored.

$$= \frac{-4 \pm \sqrt{4^2 - 4(1)(-1)}}{2(1)}$$

Let $a = 1$, $b = 4$, and $c = -1$.

$$= \frac{-4 \pm \sqrt{20}}{2}$$

Multiply and subtract under the radical: $4^2 - 4(1)(-1) = 16 - (-4) = 16 + 4 = 20$.

$$= \frac{-4 \pm 2\sqrt{5}}{2}$$

$\sqrt{20} = \sqrt{4 \cdot 5} = 2\sqrt{5}$

$$= -2 \pm \sqrt{5}$$

Divide the numerator and the denominator by 2.

The solution set is $\{-3, -2 - \sqrt{5}, -2 + \sqrt{5}\}$. The zeros of $f(x) = x^3 + 7x^2 + 11x - 3$ are -3, $-2 - \sqrt{5}$, and $-2 + \sqrt{5}$. Among these three real zeros, one zero is rational and two are irrational.

Check Point 4 Find all zeros of $f(x) = x^3 + x^2 - 5x - 2$.

If the degree of a polynomial function or equation is 4 or higher, it is often necessary to find more than one linear factor by synthetic division.

One way to speed up the process of finding the first zero is to graph the function. Any x-intercept is a zero.

3 Solve polynomial equations.

EXAMPLE 5 Solving a Polynomial Equation

Solve: $x^4 - 6x^2 - 8x + 24 = 0$.

Solution Recall that we refer to the *zeros* of a polynomial function and the *roots* of a polynomial equation. Because we are given an equation, we will use the word "roots," rather than "zeros," in the solution process. We begin by listing all possible rational roots.

$$\text{Possible rational roots} = \frac{\text{Factors of the constant term, 24}}{\text{Factors of the leading coefficient, 1}}$$
$$= \frac{\pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm24}{\pm1}$$
$$= \pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm24$$

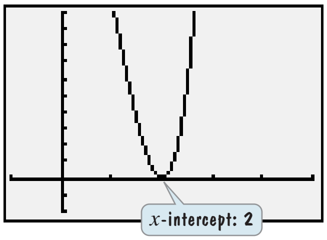


Figure 2.27 The graph of $f(x) = x^4 - 6x^2 - 8x + 24$ in a $[-1, 5, 1]$ by $[-2, 10, 1]$ viewing rectangle

Part of the graph of $f(x) = x^4 - 6x^2 - 8x + 24$ is shown in **Figure 2.27**. Because the x-intercept is 2, we will test 2 by synthetic division and show that it is a root of the given equation. Without the graph, the procedure would be to start the trial-and-error synthetic division with 1 and proceed until a zero remainder is found, as we did in Example 4.

2	1	0	−6	−8	24
		2	4	−4	−24
	1	2	−2	−12	0

Careful! $x^4 - 6x^2 - 8x + 24 = x^4 + 0x^3 - 6x^2 - 8x + 24$

The zero remainder indicates that 2 is a root of $x^4 - 6x^2 - 8x + 24 = 0$.

Now we can rewrite the given equation in factored form.

$$x^4 - 6x^2 - 8x + 24 = 0$$ This is the given equation.

$$(x-2)(x^3 + 2x^2 - 2x - 12) = 0$$ This is the result obtained from the synthetic division. The first four numbers in the bottom row, 1, 2, −2, and −12, give the coefficients of the second factor.

$$x - 2 = 0 \quad \text{or} \quad x^3 + 2x^2 - 2x - 12 = 0$$ Set each factor equal to 0.

We can use the same approach to look for rational roots of the polynomial equation $x^3 + 2x^2 - 2x - 12 = 0$, listing all possible rational roots. Without the graph in **Figure 2.27**, the procedure would be to start testing possible rational roots by trial-and-error synthetic division. However, take a second look at the graph in **Figure 2.27**. Because the graph turns around at 2, this means that 2 is a root of even multiplicity. Thus, 2 must also be a root of $x^3 + 2x^2 - 2x - 12 = 0$, confirmed by the following synthetic division.

$$\begin{array}{r|rrrr} 2 & 1 & 2 & -2 & -12 \\ & & 2 & 8 & 12 \\ \hline & 1 & 4 & 6 & 0 \end{array}$$

These are the coefficients of $x^3 + 2x^2 - 2x - 12 = 0$.

The zero remainder indicates that 2 is a root of $x^3 + 2x^2 - 2x - 12 = 0$.

Now we can solve the original equation as follows:

$$x^4 - 6x^2 - 8x + 24 = 0$$ This is the given equation.

$$(x-2)(x^3 + 2x^2 - 2x - 12) = 0$$ This factorization was obtained from the first synthetic division.

$$(x-2)(x-2)(x^2 + 4x + 6) = 0$$ This factorization was obtained from the second synthetic division. The first three numbers in the bottom row, 1, 4, and 6, give the coefficients of the third factor.

$$x - 2 = 0 \quad \text{or} \quad x - 2 = 0 \quad \text{or} \quad x^2 + 4x + 6 = 0$$ Set each factor equal to 0.

$$x = 2 \qquad\qquad x = 2.$$ Solve the linear equations.

We can use the quadratic formula to solve $x^2 + 4x + 6 = 0$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ We use the quadratic formula because $x^2 + 4x + 6$ cannot be factored.

$$= \frac{-4 \pm \sqrt{4^2 - 4(1)(6)}}{2(1)}$$ Let $a = 1$, $b = 4$, and $c = 6$.

$$= \frac{-4 \pm \sqrt{-8}}{2}$$ Multiply and subtract under the radical: $4^2 - 4(1)(6) = 16 - 24 = -8$.

$$= \frac{-4 \pm 2i\sqrt{2}}{2}$$ $\sqrt{-8} = \sqrt{4(2)(-1)} = 2i\sqrt{2}$

$$= -2 \pm i\sqrt{2}$$ Simplify.

The solution set of the original equation, $x^4 - 6x^2 - 8x + 24 = 0$, is $\{2, -2 - i\sqrt{2}, -2 + i\sqrt{2}\}$. A graphing utility does not reveal the two imaginary roots.

In Example 5, 2 is a repeated root of the equation with multiplicity 2. Counting this multiple root separately, the fourth-degree equation $x^4 - 6x^2 - 8x + 24 = 0$ has four roots: 2, 2, $-2 + i\sqrt{2}$, and $-2 - i\sqrt{2}$. The equation and its roots illustrate two general properties:

Properties of Roots of Polynomial Equations

1. If a polynomial equation is of degree n, then counting multiple roots separately, the equation has n roots.
2. If $a + bi$ is a root of a polynomial equation with real coefficients $(b \neq 0)$, then the imaginary number $a - bi$ is also a root. Imaginary roots, if they exist, occur in conjugate pairs.

Check Point 5 Solve: $x^4 - 6x^3 + 22x^2 - 30x + 13 = 0$.

The Fundamental Theorem of Algebra

The fact that a polynomial equation of degree n has n roots is a consequence of a theorem proved in 1799 by a 22-year-old student named Carl Friedrich Gauss in his doctoral dissertation. His result is called the **Fundamental Theorem of Algebra**.

Study Tip

As you read the Fundamental Theorem of Algebra, don't confuse *complex root* with *imaginary root* and conclude that every polynomial equation has at least one imaginary root. Recall that complex numbers, $a + bi$, include both real numbers $(b = 0)$ and imaginary numbers $(b \neq 0)$.

The Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree n, where $n \geq 1$, then the equation $f(x) = 0$ has at least one complex root.

Suppose, for example, that $f(x) = 0$ represents a polynomial equation of degree n. By the Fundamental Theorem of Algebra, we know that this equation has at least one complex root; we'll call it c_1. By the Factor Theorem, we know that $x - c_1$ is a factor of $f(x)$. Therefore, we obtain

$$(x - c_1)q_1(x) = 0 \quad \text{The degree of the polynomial } q_1(x) \text{ is } n - 1.$$
$$x - c_1 = 0 \quad \text{or} \quad q_1(x) = 0. \quad \text{Set each factor equal to 0.}$$

If the degree of $q_1(x)$ is at least 1, by the Fundamental Theorem of Algebra, the equation $q_1(x) = 0$ has at least one complex root. We'll call it c_2. The Factor Theorem gives us

$$q_1(x) = 0 \quad \text{The degree of } q_1(x) \text{ is } n - 1.$$
$$(x - c_2)q_2(x) = 0 \quad \text{The degree of } q_2(x) \text{ is } n - 2.$$
$$x - c_2 = 0 \quad \text{or} \quad q_2(x) = 0. \quad \text{Set each factor equal to 0.}$$

Let's see what we have up to this point and then continue the process.

$$f(x) = 0 \quad \text{This is the original polynomial equation of degree } n.$$
$$(x - c_1)q_1(x) = 0 \quad \text{This is the result from our first application of the Fundamental Theorem.}$$
$$(x - c_1)(x - c_2)q_2(x) = 0 \quad \text{This is the result from our second application of the Fundamental Theorem.}$$

By continuing this process, we will obtain the product of n linear factors. Setting each of these linear factors equal to zero results in n complex roots. Thus, if $f(x)$ is a polynomial of degree n, where $n \geq 1$, then $f(x) = 0$ has exactly n roots, where roots are counted according to their multiplicity.

4 Use the Linear Factorization Theorem to find polynomials with given zeros.

The Linear Factorization Theorem

In Example 5, we found that $x^4 - 6x^2 - 8x + 24 = 0$ has $\{2, -2 \pm i\sqrt{2}\}$ as a solution set, where 2 is a repeated root with multiplicity 2. The polynomial can be factored over the complex nonreal numbers as follows:

These are the four zeros.

$$f(x) = x^4 - 6x^2 - 8x + 24$$
$$= [x - (-2 + i\sqrt{2})][x - (-2 - i\sqrt{2})](x - 2)(x - 2).$$

These are the linear factors.

This fourth-degree polynomial has four linear factors. Just as an nth-degree polynomial equation has n roots, an nth-degree polynomial has n linear factors. This is formally stated as the **Linear Factorization Theorem**.

The Linear Factorization Theorem

If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $n \geq 1$ and $a_n \neq 0$, then

$$f(x) = a_n(x - c_1)(x - c_2)\cdots(x - c_n),$$

where $c_1, c_2, \ldots, c_n$ are complex numbers (possibly real and not necessarily distinct). In words: An nth-degree polynomial can be expressed as the product of a nonzero constant and n linear factors, where each linear factor has a leading coefficient of 1.

Many of our problems involving polynomial functions and polynomial equations dealt with the process of finding zeros and roots. The Linear Factorization Theorem enables us to reverse this process, finding a polynomial function when the zeros are given.

EXAMPLE 6 Finding a Polynomial Function with Given Zeros

Find a fourth-degree polynomial function $f(x)$ with real coefficients that has -2, 2, and i as zeros and such that $f(3) = -150$.

Solution Because i is a zero and the polynomial has real coefficients, the conjugate, $-i$, must also be a zero. We can now use the Linear Factorization Theorem.

$f(x) = a_n(x - c_1)(x - c_2)(x - c_3)(x - c_4)$ — This is the linear factorization for a fourth-degree polynomial.

$= a_n(x + 2)(x - 2)(x - i)(x + i)$ — Use the given zeros: $c_1 = -2$, $c_2 = 2$, $c_3 = i$, and, from above, $c_4 = -i$.

$= a_n(x^2 - 4)(x^2 + 1)$ — Multiply: $(x - i)(x + i) = x^2 - i^2 = x^2 - (-1) = x^2 + 1$.

$f(x) = a_n(x^4 - 3x^2 - 4)$ — Complete the multiplication.

$f(3) = a_n(3^4 - 3 \cdot 3^2 - 4) = -150$ — To find a_n, use the fact that $f(3) = -150$.

$a_n(81 - 27 - 4) = -150$ — Solve for a_n.

$50a_n = -150$ — Simplify: $81 - 27 - 4 = 50$.

$a_n = -3$ — Divide both sides by 50.

Substituting -3 for a_n in the formula for $f(x)$, we obtain

$$f(x) = -3(x^4 - 3x^2 - 4).$$

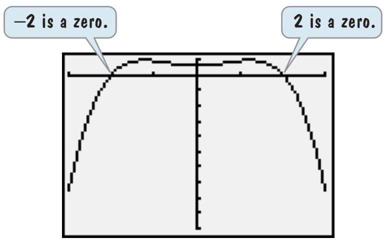
Equivalently,

$$f(x) = -3x^4 + 9x^2 + 12.$$

Technology

Graphic Connections

The graph of $f(x) = -3x^4 + 9x^2 + 12$, shown in a $[-3, 3, 1]$ by $[-200, 20, 20]$ viewing rectangle, verifies that -2 and 2 are real zeros. By tracing along the curve, we can check that $f(3) = -150$.



Check Point 6 Find a third-degree polynomial function $f(x)$ with real coefficients that has -3 and i as zeros and such that $f(1) = 8$.

5 Use Descartes's Rule of Signs.

Descartes's Rule of Signs

Because an nth-degree polynomial equation might have roots that are imaginary numbers, we should note that such an equation can have *at most* n real roots. **Descartes's Rule of Signs** provides even more specific information about the number of real zeros that a polynomial can have. The rule is based on considering *variations in sign* between consecutive coefficients. For example, the function $f(x) = 3x^7 - 2x^5 - x^4 + 7x^2 + x - 3$ has three sign changes:

$$f(x) = 3x^7 - 2x^5 - x^4 + 7x^2 + x - 3.$$

sign change — sign change — sign change

"An equation can have as many true [positive] roots as it contains changes of sign, from plus to minus or from minus to plus." René Descartes (1596–1650) in *La Géométrie* (1637)

Descartes's Rule of Signs

Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ be a polynomial with real coefficients.

1. The number of *positive real zeros* of f is either

a. the same as the number of sign changes of $f(x)$

or

b. less than the number of sign changes of $f(x)$ by a positive even integer. If $f(x)$ has only one variation in sign, then f has exactly one positive real zero.

2. The number of *negative real zeros* of f is either

a. the same as the number of sign changes of $f(-x)$

or

b. less than the number of sign changes of $f(-x)$ by a positive even integer. If $f(-x)$ has only one variation in sign, then f has exactly one negative real zero.

Study Tip

The number of real zeros given by Descartes's Rule of Signs includes rational zeros from a list of possible rational zeros, as well as irrational zeros not on the list. It does not include any imaginary zeros.

Table 2.1 illustrates what Descartes's Rule of Signs tells us about the positive real zeros of various polynomial functions.

Table 2.1 Descartes's Rule of Signs and Positive Real Zeros

Polynomial Function	Sign Changes	Conclusion
$f(x) = 3x^7 - 2x^5 - x^4 + 7x^2 + x - 3$	3	There are 3 positive real zeros. or There is $3 - 2 = 1$ positive real zero.
$f(x) = 4x^5 + 2x^4 - 3x^2 + x + 5$	2	There are 2 positive real zeros. or There are $2 - 2 = 0$ positive real zeros.
$f(x) = -7x^6 - 5x^4 + x + 9$	1	There is 1 positive real zero.

Study Tip

Be sure that the polynomial function is written in descending powers of x when counting sign changes.

EXAMPLE 7 Using Descartes's Rule of Signs

Determine the possible numbers of positive and negative real zeros of $f(x) = x^3 + 2x^2 + 5x + 4$.

Solution

1. To find possibilities for positive real zeros, count the number of sign changes in the equation for $f(x)$. Because all the coefficients are positive, there are no variations in sign. Thus, there are no positive real zeros.
2. To find possibilities for negative real zeros, count the number of sign changes in the equation for $f(-x)$. We obtain this equation by replacing x with $-x$ in the given function.

$$f(x) = x^3 + 2x^2 + 5x + 4$$

Replace x with $-x$.

$$f(-x) = (-x)^3 + 2(-x)^2 + 5(-x) + 4$$
$$= -x^3 + 2x^2 - 5x + 4$$

Now count the sign changes.

$$f(-x) = -x^3 + 2x^2 - 5x + 4$$

sign change — sign change — sign change

There are three variations in sign. The number of negative real zeros of f is either equal to the number of sign changes, 3, or is less than this number by an even integer. This means that either there are 3 negative real zeros or there is $3 - 2 = 1$ negative real zero.

What do the results of Example 7 mean in terms of solving

$$x^3 + 2x^2 + 5x + 4 = 0?$$

Without using Descartes's Rule of Signs, we list the possible rational roots as follows:

Possible rational roots

$$= \frac{\text{Factors of the constant term, 4}}{\text{Factors of the leading coefficient, 1}} = \frac{\pm 1, \pm 2, \pm 4}{\pm 1} = \pm 1, \pm 2, \pm 4.$$

However, Descartes's Rule of Signs informed us that $f(x) = x^3 + 2x^2 + 5x + 4$ has no positive real zeros. Thus, the polynomial equation $x^3 + 2x^2 + 5x + 4 = 0$ has no positive real roots. This means that we can eliminate the positive numbers from our list of possible rational roots. Possible rational roots include only -1, -2, and -4. We can use synthetic division and test the first of these three possible rational roots of $x^3 + 2x^2 + 5x + 4 = 0$ as follows:

Test -1.

-1	1	2	5	4
		-1	-1	-4
	1	1	4	0.

The zero remainder shows that -1 is a root.

By solving the equation $x^3 + 2x^2 + 5x + 4 = 0$, you will find that this equation of degree 3 has three roots. One root is -1 and the other two roots are imaginary numbers in a conjugate pair. Verify this by completing the solution process.

Check Point 7 Determine the possible numbers of positive and negative real zeros of $f(x) = x^4 - 14x^3 + 71x^2 - 154x + 120$.

Exercise Set 2.5

Practice Exercises

In Exercises 1–8, use the Rational Zero Theorem to list all possible rational zeros for each given function.

1. $f(x) = x^3 + x^2 - 4x - 4$
2. $f(x) = x^3 + 3x^2 - 6x - 8$
3. $f(x) = 3x^4 - 11x^3 - x^2 + 19x + 6$
4. $f(x) = 2x^4 + 3x^3 - 11x^2 - 9x + 15$
5. $f(x) = 4x^4 - x^3 + 5x^2 - 2x - 6$
6. $f(x) = 3x^4 - 11x^3 - 3x^2 - 6x + 8$
7. $f(x) = x^5 - x^4 - 7x^3 + 7x^2 - 12x - 12$
8. $f(x) = 4x^5 - 8x^4 - x + 2$

In Exercises 9–16,

a. *List all possible rational zeros.*

b. *Use synthetic division to test the possible rational zeros and find an actual zero.*

c. *Use the quotient from part (b) to find the remaining zeros of the polynomial function.*

9. $f(x) = x^3 + x^2 - 4x - 4$

10. $f(x) = x^3 - 2x^2 - 11x + 12$

11. $f(x) = 2x^3 - 3x^2 - 11x + 6$

12. $f(x) = 2x^3 - 5x^2 + x + 2$

13. $f(x) = x^3 + 4x^2 - 3x - 6$

14. $f(x) = 2x^3 + x^2 - 3x + 1$

15. $f(x) = 2x^3 + 6x^2 + 5x + 2$

16. $f(x) = x^3 - 4x^2 + 8x - 5$

In Exercises 17–24,

a. *List all possible rational roots.*

b. *Use synthetic division to test the possible rational roots and find an actual root.*

c. *Use the quotient from part (b) to find the remaining roots and solve the equation.*

17. $x^3 - 2x^2 - 11x + 12 = 0$

18. $x^3 - 2x^2 - 7x - 4 = 0$

19. $x^3 - 10x - 12 = 0$

20. $x^3 - 5x^2 + 17x - 13 = 0$

21. $6x^3 + 25x^2 - 24x + 5 = 0$

22. $2x^3 - 5x^2 - 6x + 4 = 0$

23. $x^4 - 2x^3 - 5x^2 + 8x + 4 = 0$

24. $x^4 - 2x^2 - 16x - 15 = 0$

In Exercises 25–32, find an nth-degree polynomial function with real coefficients satisfying the given conditions. If you are using a graphing utility, use it to graph the function and verify the real zeros and the given function value.

25. $n = 3$; 1 and $5i$ are zeros; $f(-1) = -104$

26. $n = 3$; 4 and $2i$ are zeros; $f(-1) = -50$

27. $n = 3$; -5 and $4 + 3i$ are zeros; $f(2) = 91$

28. $n = 3$; 6 and $-5 + 2i$ are zeros; $f(2) = -636$

29. $n = 4$; i and $3i$ are zeros; $f(-1) = 20$

30. $n = 4$; $-2, -\frac{1}{2}$, and i are zeros; $f(1) = 18$

31. $n = 4$; -2, 5, and $3 + 2i$ are zeros; $f(1) = -96$

32. $n = 4$; $-4, \frac{1}{3}$, and $2 + 3i$ are zeros; $f(1) = 100$

In Exercises 33–38, use Descartes's Rule of Signs to determine the possible number of positive and negative real zeros for each given function.

33. $f(x) = x^3 + 2x^2 + 5x + 4$

34. $f(x) = x^3 + 7x^2 + x + 7$

35. $f(x) = 5x^3 - 3x^2 + 3x - 1$

36. $f(x) = -2x^3 + x^2 - x + 7$

37. $f(x) = 2x^4 - 5x^3 - x^2 - 6x + 4$

38. $f(x) = 4x^4 - x^3 + 5x^2 - 2x - 6$

In Exercises 39–52, find all zeros of the polynomial function or solve the given polynomial equation. Use the Rational Zero Theorem, Descartes's Rule of Signs, and possibly the graph of the polynomial function shown by a graphing utility as an aid in obtaining the first zero or the first root.

39. $f(x) = x^3 - 4x^2 - 7x + 10$

40. $f(x) = x^3 + 12x^2 + 21x + 10$

41. $2x^3 - x^2 - 9x - 4 = 0$

42. $3x^3 - 8x^2 - 8x + 8 = 0$

43. $f(x) = x^4 - 2x^3 + x^2 + 12x + 8$

44. $f(x) = x^4 - 4x^3 - x^2 + 14x + 10$

45. $x^4 - 3x^3 - 20x^2 - 24x - 8 = 0$

46. $x^4 - x^3 + 2x^2 - 4x - 8 = 0$

47. $f(x) = 3x^4 - 11x^3 - x^2 + 19x + 6$

48. $f(x) = 2x^4 + 3x^3 - 11x^2 - 9x + 15$

49. $4x^4 - x^3 + 5x^2 - 2x - 6 = 0$

50. $3x^4 - 11x^3 - 3x^2 - 6x + 8 = 0$

51. $2x^5 + 7x^4 - 18x^2 - 8x + 8 = 0$

52. $4x^5 + 12x^4 - 41x^3 - 99x^2 + 10x + 24 = 0$

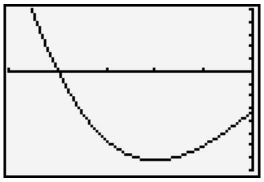
Practice Plus

Exercises 53–60, show incomplete graphs of given polynomial functions.

a. *Find all the zeros of each function.*

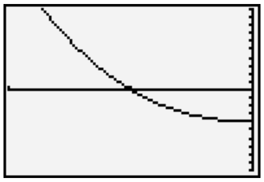
b. *Without using a graphing utility, draw a complete graph of the function.*

53. $f(x) = -x^3 + x^2 + 16x - 16$



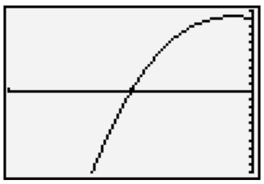
[–5, 0, 1] by [–40, 25, 5]

54. $f(x) = -x^3 + 3x^2 - 4$



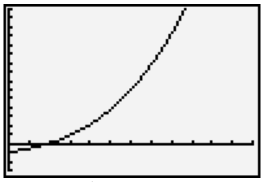
[–2, 0, 1] by [–10, 10, 1]

55. $f(x) = 4x^3 - 8x^2 - 3x + 9$



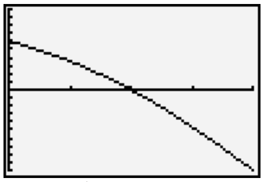
[–2, 0, 1] by [–10, 10, 1]

56. $f(x) = 3x^3 + 2x^2 + 2x - 1$



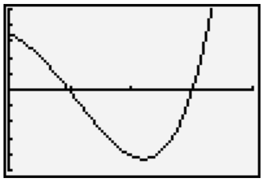
$[0, 2, \frac{1}{6}]$ by $[-3, 15, 1]$

57. $f(x) = 2x^4 - 3x^3 - 7x^2 - 8x + 6$



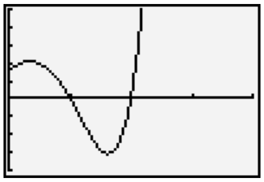
$[0, 1, \frac{1}{4}]$ by $[-10, 10, 1]$

58. $f(x) = 2x^4 + 2x^3 - 22x^2 - 18x + 36$



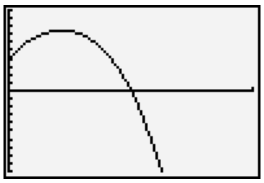
$[0, 4, 1]$ by $[-50, 50, 10]$

59. $f(x) = 3x^5 + 2x^4 - 15x^3 - 10x^2 + 12x + 8$



$[0, 4, 1]$ by $[-20, 25, 5]$

60. $f(x) = -5x^4 + 4x^3 - 19x^2 + 16x + 4$



$[0, 2, 1]$ by $[-10, 10, 1]$

Application Exercises

A popular model of carry-on luggage has a length that is 10 inches greater than its depth. Airline regulations require that the sum of the length, width, and depth cannot exceed 40 inches. These conditions, with the assumption that this sum is 40 inches, can be modeled by a function that gives the volume of the luggage, V, in cubic inches, in terms of its depth, x, in inches.

Volume = depth · length · width: 40 − (depth + length)

$$V(x) = x \cdot (x + 10) \cdot [40 - (x + x + 10)]$$

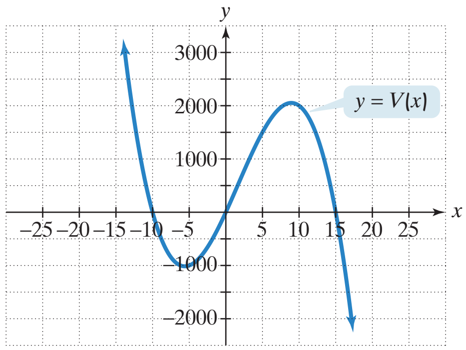
$$V(x) = x(x + 10)(30 - 2x)$$

Use function V to solve Exercises 61–62.

61. If the volume of the carry-on luggage is 2000 cubic inches, determine two possibilities for its depth. Where necessary, round to the nearest tenth of an inch.

62. If the volume of the carry-on luggage is 1500 cubic inches, determine two possibilities for its depth. Where necessary, round to the nearest tenth of an inch.

Use the graph of the function modeling the volume of the carry-on luggage to solve Exercises 63–64.



63. a. Identify your answers from Exercise 61 as points on the graph.

b. Use the graph to describe a realistic domain, x, for the volume function, where x represents the depth of the carry-on luggage.

64. a. Identify your answers from Exercise 62 as points on the graph.

b. Use the graph to describe a realistic domain, x, for the volume function, where x represents the depth of the carry-on luggage.

Writing in Mathematics

65. Describe how to find the possible rational zeros of a polynomial function.

66. How does the linear factorization of $f(x)$, that is,

$$f(x) = a_n(x - c_1)(x - c_2)\cdots(x - c_n),$$

show that a polynomial equation of degree n has n roots?

67. Describe how to use Descartes's Rule of Signs to determine the possible number of positive real zeros of a polynomial function.

68. Describe how to use Descartes's Rule of Signs to determine the possible number of negative roots of a polynomial equation.

69. Why must every polynomial equation with real coefficients of degree 3 have at least one real root?

70. Explain why the equation $x^4 + 6x^2 + 2 = 0$ has no rational roots.

71. Suppose $\frac{3}{4}$ is a root of a polynomial equation. What does this tell us about the leading coefficient and the constant term in the equation?

Technology Exercises

The equations in Exercises 72–75 have real roots that are rational. Use the Rational Zero Theorem to list all possible rational roots. Then graph the polynomial function in the given viewing rectangle to determine which possible rational roots are actual roots of the equation.

72. $2x^3 - 15x^2 + 22x + 15 = 0$; $[-1, 6, 1]$ by $[-50, 50, 10]$

73. $6x^3 - 19x^2 + 16x - 4 = 0$; $[0, 2, 1]$ by $[-3, 2, 1]$

74. $2x^4 + 7x^3 - 4x^2 - 27x - 18 = 0$; $[-4, 3, 1]$ by $[-45, 45, 15]$

75. $4x^4 + 4x^3 + 7x^2 - x - 2 = 0$; $[-2, 2, 1]$ by $[-5, 5, 1]$

76. Use Descartes's Rule of Signs to determine the possible number of positive and negative real zeros of $f(x) = 3x^4 + 5x^2 + 2$. What does this mean in terms of the graph of f? Verify your result by using a graphing utility to graph f.

77. Use Descartes's Rule of Signs to determine the possible number of positive and negative real zeros of $f(x) = x^5 - x^4 + x^3 - x^2 + x - 8$. Verify your result by using a graphing utility to graph f.

78. Write equations for several polynomial functions of odd degree and graph each function. Is it possible for the graph to have no real zeros? Explain. Try doing the same thing for polynomial functions of even degree. Now is it possible to have no real zeros?

Use a graphing utility to obtain a complete graph for each polynomial function in Exercises 79–82. Then determine the number of real zeros and the number of imaginary zeros for each function.

79. $f(x) = x^3 - 6x - 9$

80. $f(x) = 3x^5 - 2x^4 + 6x^3 - 4x^2 - 24x + 16$

81. $f(x) = 3x^4 + 4x^3 - 7x^2 - 2x - 3$

82. $f(x) = x^6 - 64$

Critical Thinking Exercises

Make Sense? *In Exercises 83–86, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

83. I've noticed that $f(-x)$ is used to explore the number of negative real zeros of a polynomial function, as well as to determine whether a function is even, odd, or neither.

84. By using the quadratic formula, I do not need to bother with synthetic division when solving polynomial equations of degree 3 or higher.

85. I'm working with a fourth-degree polynomial function with integer coefficients and zeros at 1 and $3 + \sqrt{5}$. I'm certain that $3 + \sqrt{2}$ cannot also be a zero of this function.

86. I'm working with the polynomial function $f(x) = x^4 + 3x^2 + 2$ that has four possible rational zeros but no actual rational zeros.

In Exercises 87–90, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

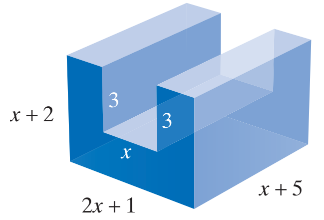
87. The equation $x^3 + 5x^2 + 6x + 1 = 0$ has one positive real root.

88. Descartes's Rule of Signs gives the exact number of positive and negative real roots for a polynomial equation.

89. Every polynomial equation of degree 3 with integer coefficients has at least one rational root.

90. Every polynomial equation of degree n has n distinct solutions.

91. If the volume of the solid shown in the figure is 208 cubic inches, find the value of x.



92. In this exercise, we lead you through the steps involved in the proof of the Rational Zero Theorem. Consider the polynomial equation

$$a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0 = 0,$$

and let $\frac{p}{q}$ be a rational root reduced to lowest terms..

a. Substitute $\frac{p}{q}$ for x in the equation and show that the equation can be written as

$$a_np^n + a_{n-1}p^{n-1}q + a_{n-2}p^{n-2}q^2 + \cdots + a_1pq^{n-1} = -a_0q^n.$$

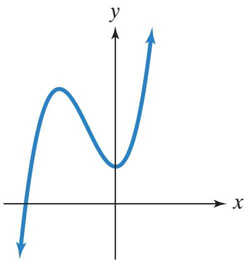
b. Why is p a factor of the left side of the equation?

c. Because p divides the left side, it must also divide the right side. However, because $\frac{p}{q}$ is reduced to lowest terms, p and q have no common factors other than -1 and 1. Because p does divide the right side and has no factors in common with q^n, what can you conclude?

d. Rewrite the equation from part (a) with all terms containing q on the left and the term that does not have a factor of q on the right. Use an argument that parallels parts (b) and (c) to conclude that q is a factor of a_n.

In Exercises 93–96, the graph of a polynomial function is given. What is the smallest degree that each polynomial could have?

93.



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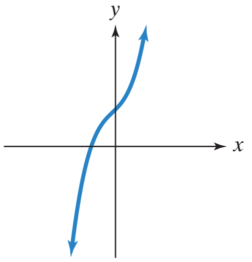

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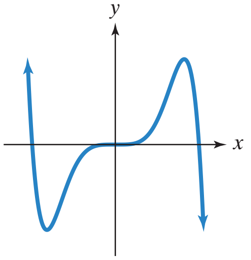

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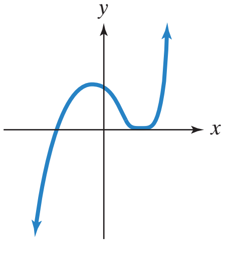
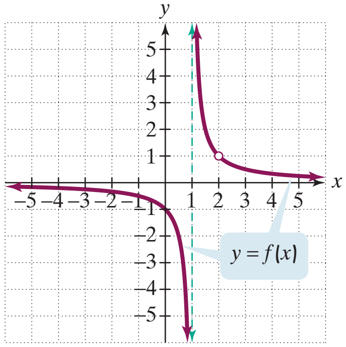


97. Explain why a polynomial function of degree 20 cannot cross the x-axis exactly once.

Preview Exercises

Exercises 98–100 will help you prepare for the material covered in the next section. Use the graph of function f to solve each exercise.



98. For what values of x is the function undefined?

99. Write the equation of the vertical asymptote, or the vertical line that the graph of f approaches but does not touch.

100. Write the equation of the horizontal asymptote, or the horizontal line that the graph of f approaches but does not touch.

Chapter 2 Mid-Chapter Check Point

What You Know: We performed operations with complex numbers and used the imaginary unit $i\left(i = \sqrt{-1}\right.$, where $\left.i^2 = -1\right)$ to represent solutions of quadratic equations with negative discriminants. Only real solutions correspond to x-intercepts. We graphed quadratic functions using vertices, intercepts, and additional points, as necessary. We learned that the vertex of $f(x) = a(x - h)^2 + k$ is (h, k) and the vertex of $f(x) = ax^2 + bx + c$ is $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$. We used the vertex to solve problems that involved minimizing or maximizing quadratic functions. We learned a number of techniques for finding the zeros of a polynomial function f of degree 3 or higher or, equivalently, finding the roots, or solutions, of the equation $f(x) = 0$. For some functions, the zeros were found by factoring $f(x)$. For other functions, we listed possible rational zeros and used synthetic division and the Factor Theorem to determine the zeros. We saw that graphs cross the x-axis at zeros of odd multiplicity and touch the x-axis and turn around at zeros of even multiplicity. We learned to graph polynomial functions using zeros, the Leading Coefficient Test, intercepts, and symmetry. We checked graphs using the fact that a polynomial function of degree n has a graph with at most $n - 1$ turning points. After finding zeros of polynomial functions, we reversed directions by using the Linear Factorization Theorem to find functions with given zeros.

In Exercises 1–6, perform the indicated operations and write the result in standard form.

1. $(6 - 2i) - (7 - i)$ **2.** $3i(2 + i)$

3. $(1 + i)(4 - 3i)$ **4.** $\dfrac{1 + i}{1 - i}$

5. $\sqrt{-75} - \sqrt{-12}$ **6.** $\left(2 - \sqrt{-3}\right)^2$

7. Solve and express solutions in standard form: $x(2x - 3) = -4$.

In Exercises 8–11, graph the given quadratic function. Give each function's domain and range.

8. $f(x) = (x - 3)^2 - 4$ **9.** $f(x) = 5 - (x + 2)^2$

10. $f(x) = -x^2 - 4x + 5$ **11.** $f(x) = 3x^2 - 6x + 1$

In Exercises 12–20, find all zeros of each polynomial function. Then graph the function.

12. $f(x) = (x - 2)^2(x + 1)^3$ **13.** $f(x) = -(x - 2)^2(x + 1)^2$

14. $f(x) = x^3 - x^2 - 4x + 4$ **15.** $f(x) = x^4 - 5x^2 + 4$

16. $f(x) = -(x + 1)^6$ **17.** $f(x) = -6x^3 + 7x^2 - 1$

18. $f(x) = 2x^3 - 2x$ **19.** $f(x) = x^3 - 2x^2 + 26x$

20. $f(x) = -x^3 + 5x^2 - 5x - 3$

In Exercises 21–26, solve each polynomial equation.

21. $x^3 - 3x + 2 = 0$

22. $6x^3 - 11x^2 + 6x - 1 = 0$

23. $(2x + 1)(3x - 2)^3(2x - 7) = 0$

24. $2x^3 + 5x^2 - 200x - 500 = 0$

25. $x^4 - x^3 - 11x^2 = x + 12$

26. $2x^4 + x^3 - 17x^2 - 4x + 6 = 0$

27. A company manufactures and sells bath cabinets. The function

$$P(x) = -x^2 + 150x - 4425$$

models the company's daily profit, $P(x)$, when x cabinets are manufactured and sold per day. How many cabinets should be manufactured and sold per day to maximize the company's profit? What is the maximum daily profit?

28. Among all pairs of numbers whose sum is -18, find a pair whose product is as large as possible. What is the maximum product?

29. The base of a triangle measures 40 inches minus twice the measure of its height. For what measure of the height does the triangle have a maximum area? What is the maximum area?

In Exercises 30–31, divide, using synthetic division if possible.

30. $(6x^4 - 3x^3 - 11x^2 + 2x + 4) \div (3x^2 - 1)$

31. $(2x^4 - 13x^3 + 17x^2 + 18x - 24) \div (x - 4)$

In Exercises 32–33, find an nth-degree polynomial function with real coefficients satisfying the given conditions.

32. $n = 3$; 1 and i are zeros; $f(-1) = 8$

33. $n = 4$; 2 (with multiplicity 2) and $3i$ are zeros; $f(0) = 36$

34. Does $f(x) = x^3 - x - 5$ have a real zero between 1 and 2?