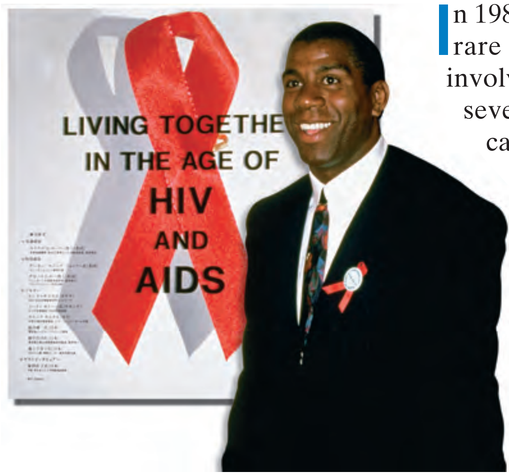
Section 2.3 Polynomial Functions and Their Graphs

Objectives

1. Identify polynomial functions.
2. Recognize characteristics of graphs of polynomial functions.
3. Determine end behavior.
4. Use factoring to find zeros of polynomial functions.
5. Identify zeros and their multiplicities.
6. Use the Intermediate Value Theorem.
7. Understand the relationship between degree and turning points.
8. Graph polynomial functions.



In 1980, U.S. doctors diagnosed 41 cases of a rare form of cancer, Kaposi's sarcoma, that involved skin lesions, pneumonia, and severe immunological deficiencies. All cases involved gay men ranging in age from 26 to 51. By the end of 2005, more than one million Americans, straight and gay, male and female, old and young, were infected with the HIV virus.

Modeling AIDS-related data and making predictions about the epidemic's havoc is serious business. **Figure 2.11** shows the number of AIDS cases diagnosed in the United States from 1983 through 2005.

AIDS Cases Diagnosed in the United States, 1983–2005

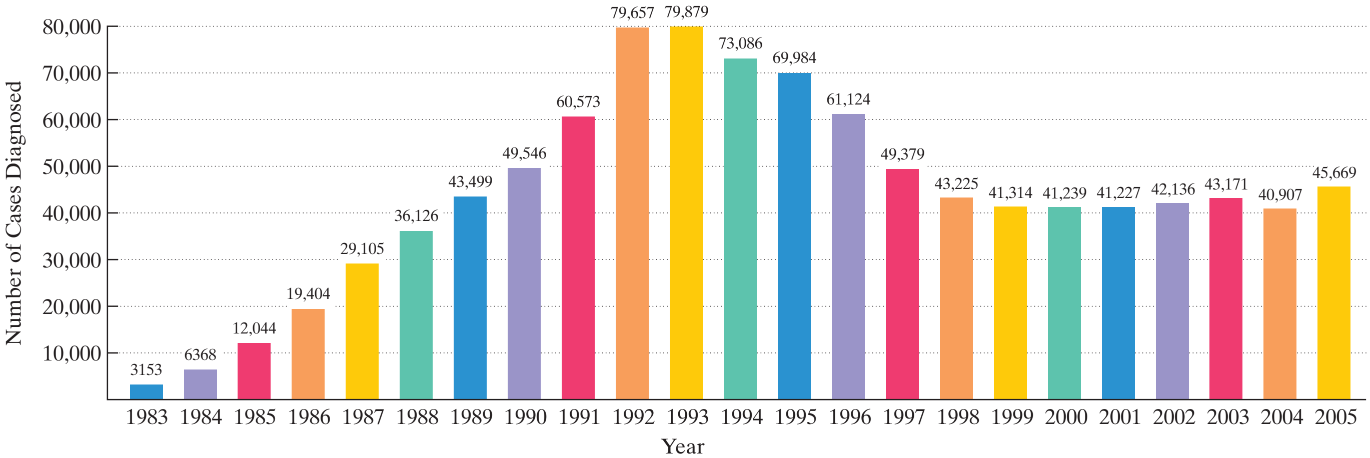


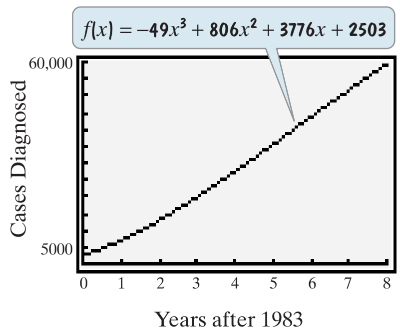
Figure 2.11
Source: Department of Health and Human Services

1 Identify polynomial functions.

Changing circumstances and unforeseen events can result in models for AIDS-related data that are not particularly useful over long periods of time. For example, the function

$$f(x) = -49x^3 + 806x^2 + 3776x + 2503$$

models the number of AIDS cases diagnosed in the United States x years after 1983. The model was obtained using a portion of the data shown in **Figure 2.11**, namely cases diagnosed from 1983 through 1991, inclusive. **Figure 2.12** shows the graph of f from 1983 through 1991. This function is an example of a *polynomial function of degree 3*.



[0, 8, 1] by [0, 60,000, 5000]

Figure 2.12 The graph of a function modeling the number of AIDS diagnoses from 1983 through 1991

Definition of a Polynomial Function

Let n be a nonnegative integer and let $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ be real numbers, with $a_n \neq 0$. The function defined by

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

is called a **polynomial function of degree n**. The number a_n, the coefficient of the variable to the highest power, is called the **leading coefficient**.

Polynomial Functions

$$f(x) = -3x^5 + \sqrt{2}x^2 + 5$$

Polynomial function of degree 5

$$g(x) = -3x^4(x-2)(x+3) = -3x^4(x^2 + x - 6) = -3x^6 - 3x^5 + 18x^4$$

Polynomial function of degree 6

Not Polynomial Functions

$$F(x) = -3\sqrt{x} + \sqrt{2}x^2 + 5 = -3x^{\frac{1}{2}} + \sqrt{2}x^2 + 5$$

The exponent on the variable is not an integer.

$$G(x) = -\frac{3}{x^2} + \sqrt{2}x^2 + 5 = -3x^{-2} + \sqrt{2}x^2 + 5$$

The exponent on the variable is not a nonnegative integer.

A constant function $f(x) = c$, where $c \neq 0$, is a polynomial function of degree 0. A linear function $f(x) = mx + b$, where $m \neq 0$, is a polynomial function of degree 1. A quadratic function $f(x) = ax^2 + bx + c$, where $a \neq 0$, is a polynomial function of degree 2. In this section, we focus on polynomial functions of degree 3 or higher.

2 Recognize characteristics of graphs of polynomial functions.

Smooth, Continuous Graphs

Polynomial functions of degree 2 or higher have graphs that are *smooth* and *continuous*. By **smooth**, we mean that the graphs contain only rounded curves with no sharp corners. By **continuous**, we mean that the graphs have no breaks and can be drawn without lifting your pencil from the rectangular coordinate system. These ideas are illustrated in **Figure 2.13**.

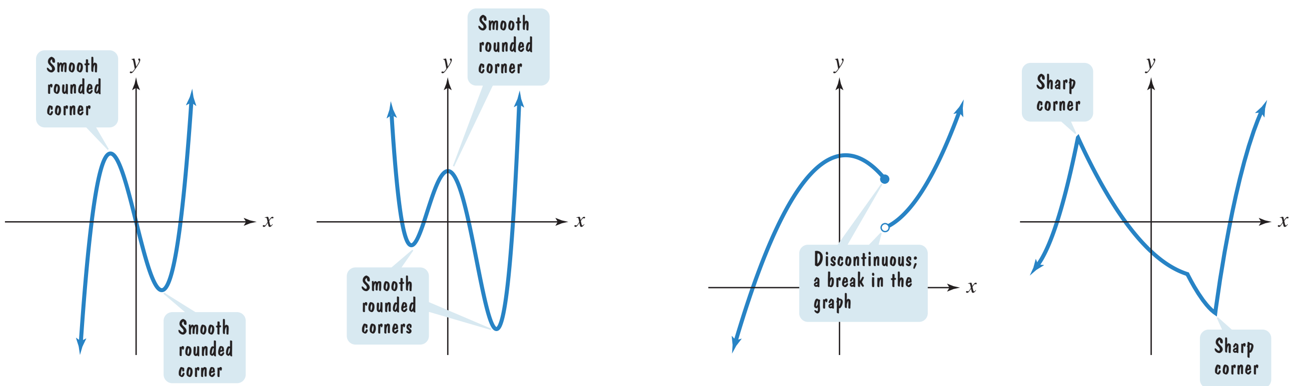


Figure 2.13 Recognizing graphs of polynomial functions

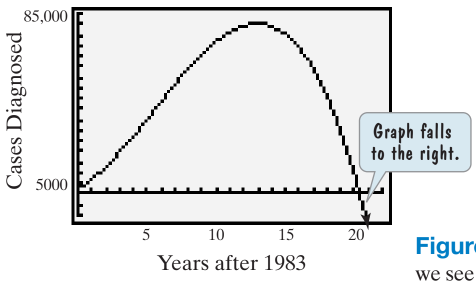
3 Determine end behavior.

End Behavior of Polynomial Functions

Figure 2.14 shows the graph of the function

$$f(x) = -49x^3 + 806x^2 + 3776x + 2503,$$

which models the number of U.S. AIDS diagnosed from 1983 through 1991. Look what happens to the graph when we extend the year up through 2005. By year 21 (2004), the values of y are negative and the function no longer models AIDS diagnoses. We've added an arrow to the graph at the far right to emphasize that it continues to decrease without bound. It is this far-right *end behavior* of the graph that makes it inappropriate for modeling AIDS cases into the future.



[0, 22, 1] by [–10,000, 85,000, 5000]

Figure 2.14 By extending the viewing rectangle, we see that y is eventually negative and the function no longer models the number of AIDS cases.

The behavior of the graph of a function to the far left or the far right is called its **end behavior**. Although the graph of a polynomial function may have intervals where it increases or decreases, the graph will eventually rise or fall without bound as it moves far to the left or far to the right.

How can you determine whether the graph of a polynomial function goes up or down at each end? The end behavior of a polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

depends upon the leading term $a_n x^n$, because when $|x|$ is large, the other terms are relatively insignificant in size. In particular, the sign of the leading coefficient, a_n, and the degree, n, of the polynomial function reveal its end behavior. In terms of end behavior, only the term of highest degree counts, as summarized by the **Leading Coefficient Test**.

The Leading Coefficient Test

As x increases or decreases without bound, the graph of the polynomial function

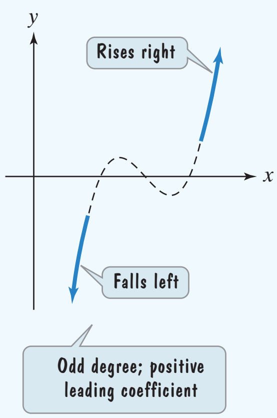
$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0 \quad (a_n \neq 0)$$

eventually rises or falls. In particular,

1. For n odd:

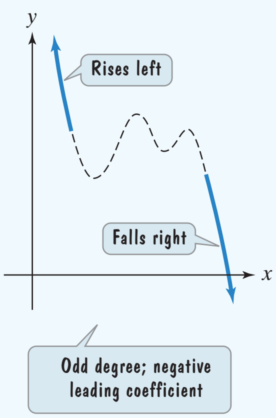
If the leading coefficient is positive, the graph falls to the left and rises to the right. (↙, ↗)

$a_n > 0$



If the leading coefficient is negative, the graph rises to the left and falls to the right. (↖, ↘)

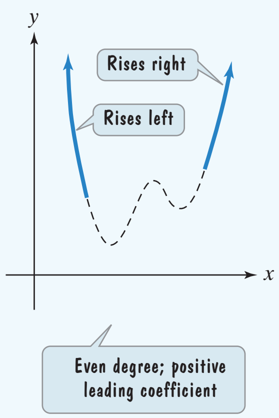
$a_n < 0$



2. For n even:

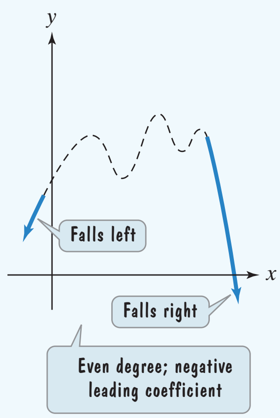
If the leading coefficient is positive, the graph rises to the left and rises to the right. (↖, ↗)

$a_n > 0$



If the leading coefficient is negative, the graph falls to the left and falls to the right. (↙, ↘)

$a_n < 0$



Study Tip

Odd-degree polynomial functions have graphs with opposite behavior at each end. Even-degree polynomial functions have graphs with the same behavior at each end.

EXAMPLE 1 Using the Leading Coefficient Test

Use the Leading Coefficient Test to determine the end behavior of the graph of

$$f(x) = x^3 + 3x^2 - x - 3.$$

Solution We begin by identifying the sign of the leading coefficient and the degree of the polynomial.

$$f(x) = x^3 + 3x^2 - x - 3$$

The leading coefficient, 1, is positive.

The degree of the polynomial, 3, is odd.

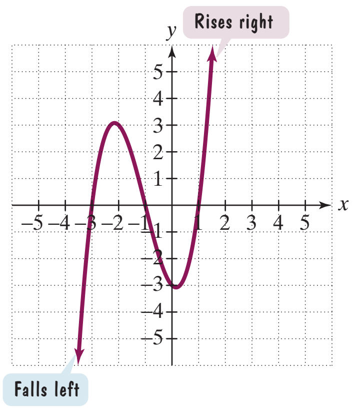


Figure 2.15 The graph of $f(x) = x^3 + 3x^2 - x - 3$

The degree of the function f is 3, which is odd. Odd-degree polynomial functions have graphs with opposite behavior at each end. The leading coefficient, 1, is positive. Thus, the graph falls to the left and rises to the right (↙, ↗). The graph of f is shown in **Figure 2.15**.

Check Point 1 Use the Leading Coefficient Test to determine the end behavior of the graph of $f(x) = x^4 - 4x^2$.

EXAMPLE 2 Using the Leading Coefficient Test

Use the Leading Coefficient Test to determine the end behavior of the graph of

$$f(x) = -4x^3(x - 1)^2(x + 5).$$

Solution Although the equation for f is in factored form, it is not necessary to multiply to determine the degree of the function.

$$f(x) = -4x^3(x - 1)^2(x + 5)$$

Degree of this factor is 3. | Degree of this factor is 2. | Degree of this factor is 1.

When multiplying exponential expressions with the same base, we add the exponents. This means that the degree of f is $3 + 2 + 1$, or 6, which is even. Even-degree polynomial functions have graphs with the same behavior at each end. Without multiplying out, you can see that the leading coefficient is −4, which is negative. Thus, the graph of f falls to the left and falls to the right (↙, ↘).

Check Point 2 Use the Leading Coefficient Test to determine the end behavior of the graph of

$$f(x) = 2x^3(x - 1)(x + 5).$$

EXAMPLE 3 Using the Leading Coefficient Test

Use end behavior to explain why

$$f(x) = -49x^3 + 806x^2 + 3776x + 2503$$

is only an appropriate model for AIDS diagnoses for a limited time period.

Solution We begin by identifying the sign of the leading coefficient and the degree of the polynomial.

$$f(x) = -49x^3 + 806x^2 + 3776x + 2503$$

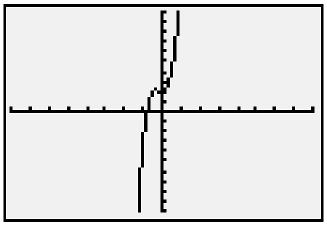
The leading coefficient, −49, is negative. | The degree of the polynomial, 3, is odd.

The degree of f is 3, which is odd. Odd-degree polynomial functions have graphs with opposite behavior at each end. The leading coefficient, −49, is negative. Thus, the graph rises to the left and falls to the right (↖, ↘). The fact that the graph falls to the right indicates that at some point the number of AIDS diagnoses will be negative, an impossibility. If a function has a graph that decreases without bound over time, it will not be capable of modeling nonnegative phenomena over long time periods. Model breakdown will eventually occur.

Check Point 3 The polynomial function

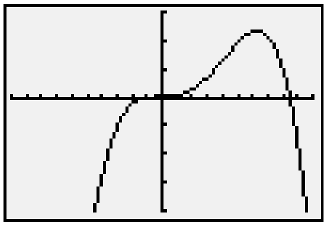
$$f(x) = -0.27x^3 + 9.2x^2 - 102.9x + 400$$

models the ratio of students to computers in U.S. public schools x years after 1980. Use end behavior to determine whether this function could be an appropriate model for computers in the classroom well into the twenty-first century. Explain your answer.



[−8, 8, 1] by [−10, 10, 1]

Figure 2.16



[−10, 10, 1] by [−1000, 750, 250]

Figure 2.17

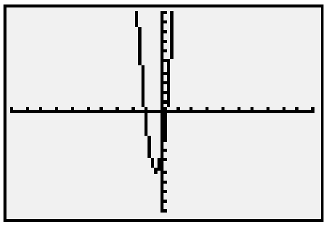


Figure 2.18

If you use a graphing utility to graph a polynomial function, it is important to select a viewing rectangle that accurately reveals the graph's end behavior. If the viewing rectangle, or window, is too small, it may not accurately show a complete graph with the appropriate end behavior.

EXAMPLE 4 Using the Leading Coefficient Test

The graph of $f(x) = -x^4 + 8x^3 + 4x^2 + 2$ was obtained with a graphing utility using a $[-8, 8, 1]$ by $[-10, 10, 1]$ viewing rectangle. The graph is shown in **Figure 2.16**. Is this a complete graph that shows the end behavior of the function?

Solution We begin by identifying the sign of the leading coefficient and the degree of the polynomial.

$$f(x) = -x^4 + 8x^3 + 4x^2 + 2$$

The leading coefficient, −1, is negative.

The degree of the polynomial, 4, is even.

The degree of f is 4, which is even. Even-degree polynomial functions have graphs with the same behavior at each end. The leading coefficient, −1, is negative. Thus, the graph should fall to the left and fall to the right (↙, ↘). The graph in **Figure 2.16** is falling to the left, but it is not falling to the right. Therefore, the graph is not complete enough to show end behavior. A more complete graph of the function is shown in a larger viewing rectangle in **Figure 2.17**.

Check Point 4 The graph of $f(x) = x^3 + 13x^2 + 10x - 4$ is shown in a standard viewing rectangle in **Figure 2.18**. Use the Leading Coefficient Test to determine whether this is a complete graph that shows the end behavior of the function. Explain your answer.

4 Use factoring to find zeros of polynomial functions.

Zeros of Polynomial Functions

If f is a polynomial function, then the values of x for which $f(x)$ is equal to 0 are called the **zeros** of f. These values of x are the **roots**, or **solutions**, of the polynomial equation $f(x) = 0$. Each real root of the polynomial equation appears as an x-intercept of the graph of the polynomial function.

EXAMPLE 5 Finding Zeros of a Polynomial Function

Find all zeros of $f(x) = x^3 + 3x^2 - x - 3$.

Solution By definition, the zeros are the values of x for which $f(x)$ is equal to 0. Thus, we set $f(x)$ equal to 0:

$$f(x) = x^3 + 3x^2 - x - 3 = 0.$$

We solve the polynomial equation $x^3 + 3x^2 - x - 3 = 0$ for x as follows:

$$x^3 + 3x^2 - x - 3 = 0$$ This is the equation needed to find the function's zeros.

$$x^2(x + 3) - 1(x + 3) = 0$$ Factor x^2 from the first two terms and −1 from the last two terms.

$$(x + 3)(x^2 - 1) = 0$$ A common factor of $x + 3$ is factored from the expression.

$$x + 3 = 0 \quad \text{or} \quad x^2 - 1 = 0$$ Set each factor equal to 0.

$$x = -3 \qquad x^2 = 1$$ Solve for x.

$$x = \pm 1$$ Remember that if $x^2 = d$, then $x = \pm\sqrt{d}$.

The zeros of f are −3, −1, and 1. The graph of f in **Figure 2.19** shows that each zero is an x-intercept. The graph passes through the points $(-3, 0)$, $(-1, 0)$, and $(1, 0)$.

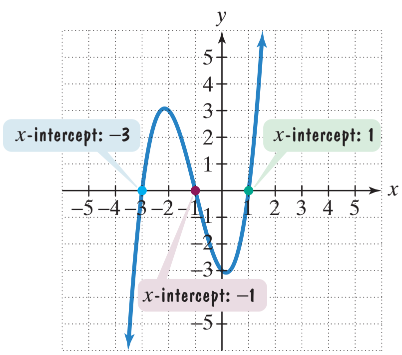


Figure 2.19

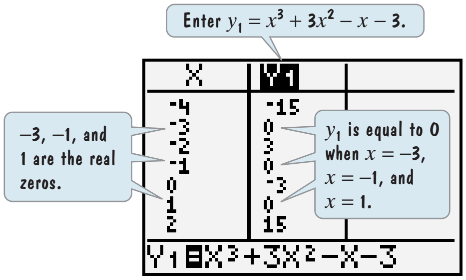
Technology

Graphic and Numeric Connections

A graphing utility can be used to verify that −3, −1, and 1 are the three real zeros of $f(x) = x^3 + 3x^2 - x - 3$.

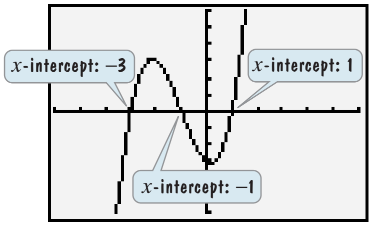
Numeric Check

Display a table for the function.



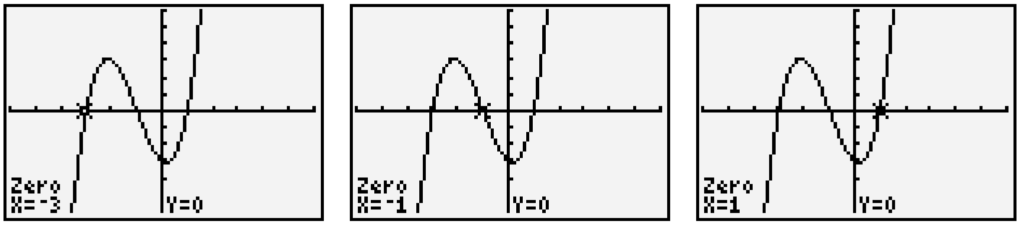
Graphic Check

Display a graph for the function. The x-intercepts indicate that −3, −1, and 1 are the real zeros.



[−6, 6, 1] by [−6, 6, 1]

The utility's ZERO feature on the graph of f also verifies that −3, −1, and 1 are the function's real zeros.



Check Point 5 Find all zeros of $f(x) = x^3 + 2x^2 - 4x - 8$.

EXAMPLE 6 Finding Zeros of a Polynomial Function

Find all zeros of $f(x) = -x^4 + 4x^3 - 4x^2$.

Solution We find the zeros of f by setting $f(x)$ equal to 0 and solving the resulting equation.

$$-x^4 + 4x^3 - 4x^2 = 0 \quad \text{We now have a polynomial equation.}$$

$$x^4 - 4x^3 + 4x^2 = 0 \quad \text{Multiply both sides by } -1. \text{ This step is optional.}$$

$$x^2(x^2 - 4x + 4) = 0 \quad \text{Factor out } x^2.$$

$$x^2(x-2)^2 = 0 \quad \text{Factor completely.}$$

$$x^2 = 0 \quad \text{or} \quad (x-2)^2 = 0 \quad \text{Set each factor equal to 0.}$$

$$x = 0 \qquad\qquad x = 2 \quad \text{Solve for } x.$$

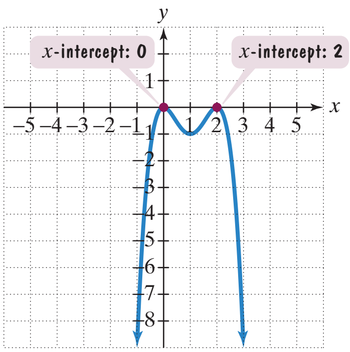


Figure 2.20 The zeros of $f(x) = -x^4 + 4x^3 - 4x^2$, namely 0 and 2, are the x-intercepts for the graph of f.

The zeros of $f(x) = -x^4 + 4x^3 - 4x^2$ are 0 and 2. The graph of f, shown in **Figure 2.20**, has x-intercepts at 0 and 2. The graph passes through the points (0, 0) and (2, 0).

Check Point 6 Find all zeros of $f(x) = x^4 - 4x^2$.

5 Identify zeros and their multiplicities.

Multiplicities of Zeros

We can use the results of factoring to express a polynomial as a product of factors. For instance, in Example 6, we can use our factoring to express the function's equation as follows:

$$f(x) = -x^4 + 4x^3 - 4x^2 = -(x^4 - 4x^3 + 4x^2) = -x^2(x-2)^2.$$

The factor x occurs twice: $x^2 = x \cdot x$.

The factor $(x-2)$ occurs twice: $(x-2)^2 = (x-2)(x-2)$.

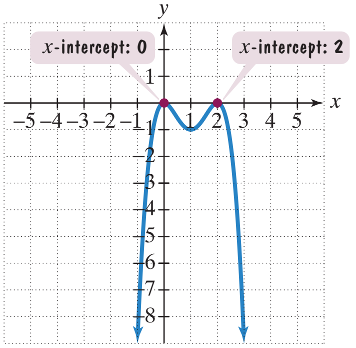


Figure 2.20 (repeated) The graph of $f(x) = -x^4 + 4x^3 - 4x^2$

Notice that each factor occurs twice. In factoring the equation for the polynomial function f, if the same factor $x - r$ occurs k times, but not $k + 1$ times, we call r a **zero with multiplicity k**. For the polynomial function

$$f(x) = -x^2(x-2)^2,$$

0 and 2 are both zeros with multiplicity 2.

Multiplicity provides another connection between zeros and graphs. The multiplicity of a zero tells us whether the graph of a polynomial function touches the x-axis at the zero and turns around, or if the graph crosses the x-axis at the zero. For example, look again at the graph of $f(x) = -x^4 + 4x^3 - 4x^2$ in **Figure 2.20**. Each zero, 0 and 2, is a zero with multiplicity 2. The graph of f touches, but does not cross, the x-axis at each of these zeros of even multiplicity. By contrast, a graph crosses the x-axis at zeros of odd multiplicity.

Multiplicity and x-Intercepts

If r is a zero of **even multiplicity**, then the graph **touches** the x-axis **and turns around** at r. If r is a zero of **odd multiplicity**, then the graph **crosses** the x-axis at r. Regardless of whether the multiplicity of a zero is even or odd, graphs tend to flatten out near zeros with multiplicity greater than one.

If a polynomial function's equation is expressed as a product of linear factors, we can quickly identify zeros and their multiplicities.

EXAMPLE 7 Finding Zeros and Their Multiplicities

Find the zeros of $f(x) = \frac{1}{2}(x+1)(2x-3)^2$ and give the multiplicity of each zero. State whether the graph crosses the x-axis or touches the x-axis and turns around at each zero.

Solution We find the zeros of f by setting $f(x)$ equal to 0:

$$\tfrac{1}{2}(x+1)(2x-3)^2 = 0.$$

Set each variable factor equal to 0.

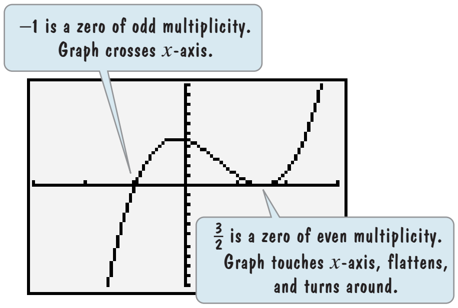
$x + 1 = 0$, $x = -1$

$2x - 3 = 0$, $x = \frac{3}{2}$

$$\tfrac{1}{2}(x+1)^1(2x-3)^2 = 0$$

This exponent is 1. Thus, the multiplicity of −1 is 1.

This exponent is 2. Thus, the multiplicity of $\frac{3}{2}$ is 2.



[−3, 3, 1] by [−10, 10, 1]

Figure 2.21 The graph of $f(x) = \frac{1}{2}(x+1)(2x-3)^2$

The zeros of $f(x) = \frac{1}{2}(x+1)(2x-3)^2$ are −1, with multiplicity 1, and $\frac{3}{2}$, with multiplicity 2. Because the multiplicity of −1 is odd, the graph crosses the x-axis at this zero. Because the multiplicity of $\frac{3}{2}$ is even, the graph touches the x-axis and turns around at this zero. These relationships are illustrated by the graph of f in **Figure 2.21**.

Check Point 7 Find the zeros of $f(x) = -4\left(x + \frac{1}{2}\right)^2(x - 5)^3$ and give the multiplicity of each zero. State whether the graph crosses the x-axis or touches the x-axis and turns around at each zero.

6 Use the Intermediate Value Theorem.

The Intermediate Value Theorem

The *Intermediate Value Theorem* tells us of the existence of real zeros. The idea behind the theorem is illustrated in **Figure 2.22**. The figure shows that if $(a, f(a))$ lies below the x-axis and $(b, f(b))$ lies above the x-axis, the smooth, continuous graph of the polynomial function f must cross the x-axis at some value c between a and b. This value is a real zero for the function.

These observations are summarized in the **Intermediate Value Theorem**.

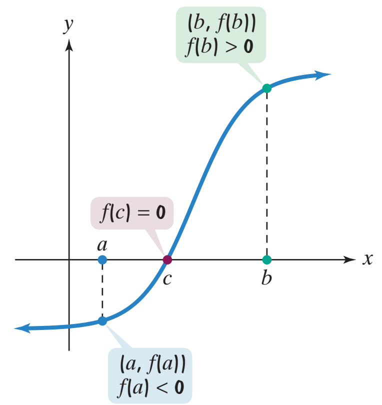


Figure 2.22 The graph must cross the x-axis at some value between a and b.

The Intermediate Value Theorem for Polynomial Functions

Let f be a polynomial function with real coefficients. If $f(a)$ and $f(b)$ have opposite signs, then there is at least one value of c between a and b for which $f(c) = 0$. Equivalently, the equation $f(x) = 0$ has at least one real root between a and b.

EXAMPLE 8 Using the Intermediate Value Theorem

Show that the polynomial function $f(x) = x^3 - 2x - 5$ has a real zero between 2 and 3.

Solution Let us evaluate f at 2 and at 3. If $f(2)$ and $f(3)$ have opposite signs, then there is at least one real zero between 2 and 3. Using $f(x) = x^3 - 2x - 5$, we obtain

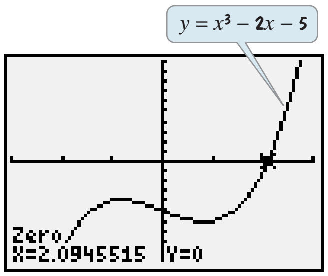
$$f(2) = 2^3 - 2 \cdot 2 - 5 = 8 - 4 - 5 = -1$$

$f(2)$ is negative.

and

$$f(3) = 3^3 - 2 \cdot 3 - 5 = 27 - 6 - 5 = 16.$$

$f(3)$ is positive.



[−3, 3, 1] by [−10, 10, 1]

Figure 2.23

Because $f(2) = -1$ and $f(3) = 16$, the sign change shows that the polynomial function has a real zero between 2 and 3. This zero is actually irrational and is approximated using a graphing utility's ZERO feature as 2.0945515 in **Figure 2.23**.

Check Point 8 Show that the polynomial function $f(x) = 3x^3 - 10x + 9$ has a real zero between -3 and -2.

7 Understand the relationship between degree and turning points.

Turning Points of Polynomial Functions

The graph of $f(x) = x^5 - 6x^3 + 8x + 1$ is shown in **Figure 2.24** on the next page. The graph has four smooth **turning points**.

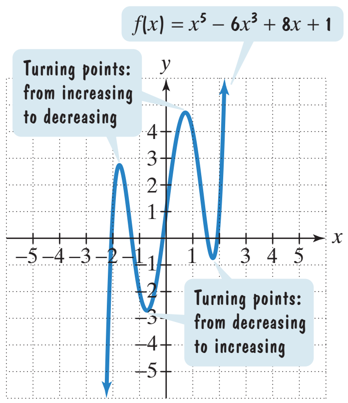


Figure 2.24 Graph with four turning points

At each turning point in **Figure 2.24**, the graph changes direction from increasing to decreasing or vice versa. The given equation has 5 as its greatest exponent and is therefore a polynomial function of degree 5. Notice that the graph has four turning points. In general, **if f is a polynomial function of degree n, then the graph of f has at most $n - 1$ turning points**.

Figure 2.24 illustrates that the y-coordinate of each turning point is either a relative maximum or a relative minimum of f. Without the aid of a graphing utility or a knowledge of calculus, it is difficult and often impossible to locate turning points of polynomial functions with degrees greater than 2. If necessary, test values can be taken between the x-intercepts to get a general idea of how high the graph rises or how low the graph falls. For the purpose of graphing in this section, a general estimate is sometimes appropriate and necessary.

8 Graph polynomial functions.

A Strategy for Graphing Polynomial Functions

Here's a general strategy for graphing a polynomial function. A graphing utility is a valuable complement, but not a necessary component, to this strategy. If you are using a graphing utility, some of the steps listed in the following box will help you to select a viewing rectangle that shows the important parts of the graph.

Graphing a Polynomial Function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0,\ a_n \neq 0$$

1. Use the Leading Coefficient Test to determine the graph's end behavior.
2. Find x-intercepts by setting $f(x) = 0$ and solving the resulting polynomial equation. If there is an x-intercept at r as a result of $(x - r)^k$ in the complete factorization of $f(x)$, then
   - **a.** If k is even, the graph touches the x-axis at r and turns around.
   - **b.** If k is odd, the graph crosses the x-axis at r.
   - **c.** If $k > 1$, the graph flattens out near $(r, 0)$.
3. Find the y-intercept by computing $f(0)$.
4. Use symmetry, if applicable, to help draw the graph:
   - **a.** y-axis symmetry: $f(-x) = f(x)$
   - **b.** Origin symmetry: $f(-x) = -f(x)$.
5. Use the fact that the maximum number of turning points of the graph is $n - 1$, where n is the degree of the polynomial function, to check whether it is drawn correctly.

Study Tip

Remember that, without calculus, it is often impossible to give the exact location of turning points. However, you can obtain additional points satisfying the function to estimate how high the graph rises or how low it falls. To obtain these points, use values of x between (and to the left and right of) the x-intercepts.

EXAMPLE 9 Graphing a Polynomial Function

Graph: $f(x) = x^4 - 2x^2 + 1$.

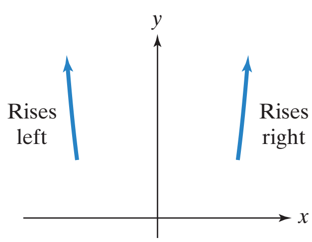
Solution

Step 1 Determine end behavior. Identify the sign of a_n, the leading coefficient, and the degree, n, of the polynomial function.

$$f(x) = x^4 - 2x^2 + 1$$

The leading coefficient, 1, is positive.

The degree of the polynomial function, 4, is even.

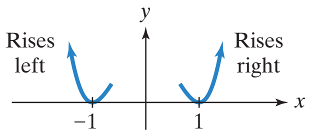


Because the degree, 4, is even, the graph has the same behavior at each end. The leading coefficient, 1, is positive. Thus, the graph rises to the left and rises to the right.

Step 2 Find x-intercepts (zeros of the function) by setting $f(x) = 0$.

$$x^4 - 2x^2 + 1 = 0 \quad \text{Set } f(x) \text{ equal to 0.}$$
$$(x^2 - 1)(x^2 - 1) = 0 \quad \text{Factor.}$$
$$(x + 1)(x - 1)(x + 1)(x - 1) = 0 \quad \text{Factor completely.}$$
$$(x + 1)^2(x - 1)^2 = 0 \quad \text{Express the factorization in a more compact form.}$$
$$(x + 1)^2 = 0 \quad \text{or} \quad (x - 1)^2 = 0 \quad \text{Set each factorization equal to 0.}$$
$$x = -1 \qquad x = 1 \quad \text{Solve for x.}$$

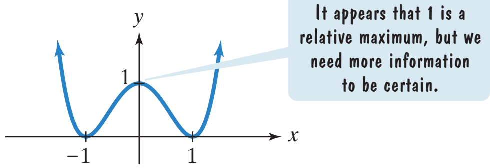
We see that -1 and 1 are both repeated zeros with multiplicity 2. Because of the even multiplicity, the graph touches the x-axis at -1 and 1 and turns around. Furthermore, the graph tends to flatten out near these zeros with multiplicity greater than one.



Step 3 Find the y-intercept by computing $f(0)$. We use $f(x) = x^4 - 2x^2 + 1$ and compute $f(0)$.

$$f(0) = 0^4 - 2 \cdot 0^2 + 1 = 1$$

There is a y-intercept at 1, so the graph passes through $(0, 1)$.



Step 4 Use possible symmetry to help draw the graph. Our partial graph suggests y-axis symmetry. Let's verify this by finding $f(-x)$.

$$f(x) = x^4 - 2x^2 + 1$$

Replace x with $-x$.

$$f(-x) = (-x)^4 - 2(-x)^2 + 1 = x^4 - 2x^2 + 1$$

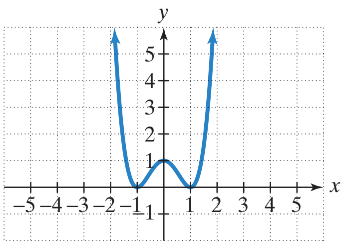


Figure 2.25 The graph of $f(x) = x^4 - 2x^2 + 1$

Because $f(-x) = f(x)$, the graph of f is symmetric with respect to the y-axis. **Figure 2.25** shows the graph of $f(x) = x^4 - 2x^2 + 1$.

Step 5 Use the fact that the maximum number of turning points of the graph is $n - 1$ to check whether it is drawn correctly. Because $n = 4$, the maximum number of turning points is $4 - 1$, or 3. Because the graph in **Figure 2.25** has three turning points, we have not violated the maximum number possible. Can you see how this verifies that 1 is indeed a relative maximum and $(0, 1)$ is a turning point? If the graph rose above 1 on either side of $x = 0$, it would have to rise above 1 on the other side as well because of symmetry. This would require additional turning points to smoothly curve back to the x-intercepts. The graph already has three turning points, which is the maximum number for a fourth-degree polynomial function.

Check Point 9 Use the five-step strategy to graph $f(x) = x^3 - 3x^2$.

Exercise Set 2.3

Practice Exercises

In Exercises 1–10, determine which functions are polynomial functions. For those that are, identify the degree.

1. $f(x) = 5x^2 + 6x^3$ **2.** $f(x) = 7x^2 + 9x^4$

3. $g(x) = 7x^5 - \pi x^3 + \frac{1}{5}x$ **4.** $g(x) = 6x^7 + \pi x^5 + \frac{2}{3}x$

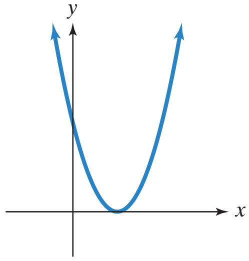
5. $h(x) = 7x^3 + 2x^2 + \frac{1}{x}$ **6.** $h(x) = 8x^3 - x^2 + \frac{2}{x}$

7. $f(x) = x^{\frac{1}{2}} - 3x^2 + 5$ **8.** $f(x) = x^{\frac{1}{3}} - 4x^2 + 7$

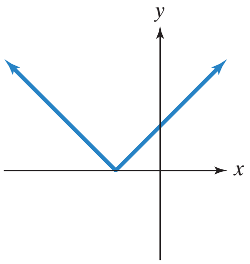
9. $f(x) = \frac{x^2 + 7}{x^3}$ **10.** $f(x) = \frac{x^2 + 7}{3}$

In Exercises 11–14, identify which graphs are not those of polynomial functions.

11.



12.



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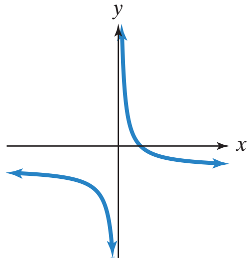
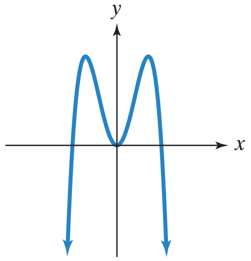


14.

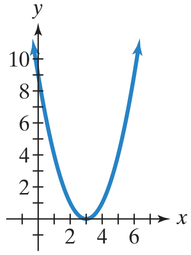


In Exercises 15–18, use the Leading Coefficient Test to determine the end behavior of the graph of the given polynomial function. Then use this end behavior to match the polynomial function with its graph. [The graphs are labeled (a) through (d).]

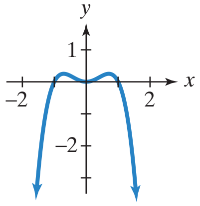
15. $f(x) = -x^4 + x^2$ **16.** $f(x) = x^3 - 4x^2$

17. $f(x) = (x - 3)^2$ **18.** $f(x) = -x^3 - x^2 + 5x - 3$

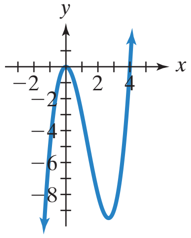
a.



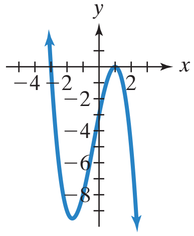
b.



c.



d.



In Exercises 19–24, use the Leading Coefficient Test to determine the end behavior of the graph of the polynomial function.

19. $f(x) = 5x^3 + 7x^2 - x + 9$

20. $f(x) = 11x^3 - 6x^2 + x + 3$

21. $f(x) = 5x^4 + 7x^2 - x + 9$

22. $f(x) = 11x^4 - 6x^2 + x + 3$

23. $f(x) = -5x^4 + 7x^2 - x + 9$

24. $f(x) = -11x^4 - 6x^2 + x + 3$

In Exercises 25–32, find the zeros for each polynomial function and give the multiplicity for each zero. State whether the graph crosses the x-axis, or touches the x-axis and turns around, at each zero.

25. $f(x) = 2(x - 5)(x + 4)^2$

26. $f(x) = 3(x + 5)(x + 2)^2$

27. $f(x) = 4(x - 3)(x + 6)^3$

28. $f(x) = -3\left(x + \frac{1}{2}\right)(x - 4)^3$

29. $f(x) = x^3 - 2x^2 + x$

30. $f(x) = x^3 + 4x^2 + 4x$

31. $f(x) = x^3 + 7x^2 - 4x - 28$

32. $f(x) = x^3 + 5x^2 - 9x - 45$

In Exercises 33–40, use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers.

33. $f(x) = x^3 - x - 1$; between 1 and 2

34. $f(x) = x^3 - 4x^2 + 2$; between 0 and 1

35. $f(x) = 2x^4 - 4x^2 + 1$; between −1 and 0

36. $f(x) = x^4 + 6x^3 - 18x^2$; between 2 and 3

37. $f(x) = x^3 + x^2 - 2x + 1$; between −3 and −2

38. $f(x) = x^5 - x^3 - 1$; between 1 and 2

39. $f(x) = 3x^3 - 10x + 9$; between −3 and −2

40. $f(x) = 3x^3 - 8x^2 + x + 2$; between 2 and 3

In Exercises 41–64,

- **a.** *Use the Leading Coefficient Test to determine the graph's end behavior.*
- **b.** *Find the x-intercepts. State whether the graph crosses the x-axis, or touches the x-axis and turns around, at each intercept.*
- **c.** *Find the y-intercept.*
- **d.** *Determine whether the graph has y-axis symmetry, origin symmetry, or neither.*
- **e.** *If necessary, find a few additional points and graph the function. Use the maximum number of turning points to check whether it is drawn correctly.*

41. $f(x) = x^3 + 2x^2 - x - 2$ **42.** $f(x) = x^3 + x^2 - 4x - 4$

43. $f(x) = x^4 - 9x^2$ **44.** $f(x) = x^4 - x^2$

45. $f(x) = -x^4 + 16x^2$ **46.** $f(x) = -x^4 + 4x^2$

47. $f(x) = x^4 - 2x^3 + x^2$

48. $f(x) = x^4 - 6x^3 + 9x^2$

49. $f(x) = -2x^4 + 4x^3$

50. $f(x) = -2x^4 + 2x^3$

51. $f(x) = 6x^3 - 9x - x^5$

52. $f(x) = 6x - x^3 - x^5$

53. $f(x) = 3x^2 - x^3$

54. $f(x) = \frac{1}{2} - \frac{1}{2}x^4$

55. $f(x) = -3(x - 1)^2(x^2 - 4)$

56. $f(x) = -2(x - 4)^2(x^2 - 25)$

57. $f(x) = x^2(x - 1)^3(x + 2)$

58. $f(x) = x^3(x + 2)^2(x + 1)$

59. $f(x) = -x^2(x - 1)(x + 3)$

60. $f(x) = -x^2(x + 2)(x - 2)$

61. $f(x) = -2x^3(x - 1)^2(x + 5)$

62. $f(x) = -3x^3(x - 1)^2(x + 3)$

63. $f(x) = (x - 2)^2(x + 4)(x - 1)$

64. $f(x) = (x + 3)(x + 1)^3(x + 4)$

Practice Plus

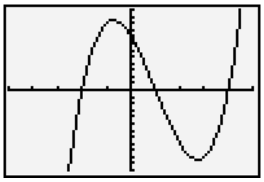
In Exercises 65–72, complete graphs of polynomial functions whose zeros are integers are shown.

a. *Find the zeros and state whether the multiplicity of each zero is even or odd.*

b. *Write an equation, expressed as the product of factors, of a polynomial function that might have each graph. Use a leading coefficient of 1 or −1, and make the degree of f as small as possible.*

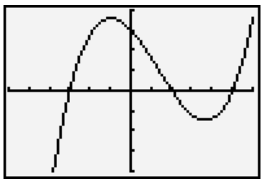
c. *Use both the equation in part (b) and the graph to find the y-intercept.*

65.



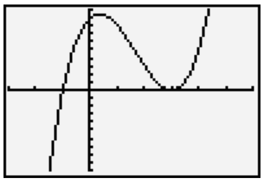
[−5, 5, 1] by [−12, 12, 1]

66.



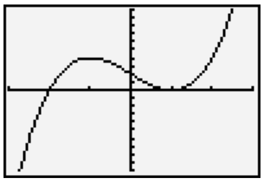
[−6, 6, 1] by [−40, 40, 10]

67.



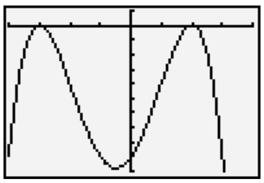
[−3, 6, 1] by [−10, 10, 1]

68.



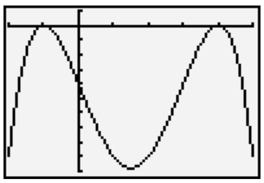
[−3, 3, 1] by [−10, 10, 1]

69.



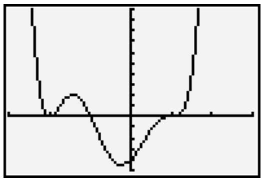
[−4, 4, 1] by [−40, 4, 4]

70.



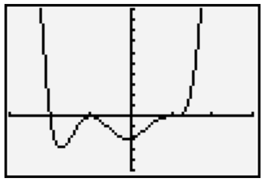
[−2, 5, 1] by [−40, 4, 4]

71.



[−3, 3, 1] by [−5, 10, 1]

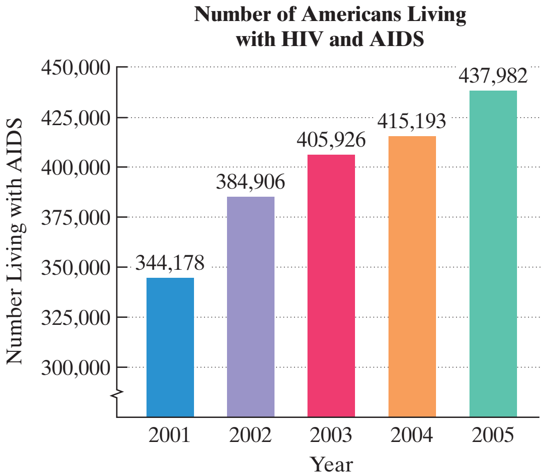
72.



[−3, 3, 1] by [−5, 10, 1]

Application Exercises

The bar graph shows the number of Americans living with HIV and AIDS from 2001 through 2005.



Source: Department of Health and Human Services

The data in the bar graph can be modeled by the following second- and third-degree polynomial functions:

Number living with HIV and AIDS x years after 2000

$$f(x) = -3402x^2 + 42{,}203x + 308{,}453$$
$$g(x) = 2769x^3 - 28{,}324x^2 + 107{,}555x + 261{,}931.$$

Use these functions to solve Exercises 73–74.

73. a. Use both functions to find the number of Americans living with HIV and AIDS in 2003. Which function provides a better description for the actual number shown in the bar graph?

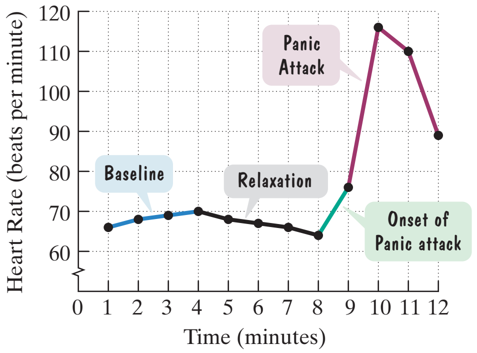
b. Consider the function from part (a) that serves as a better model for 2003. Use the Leading Coefficient Test to determine the end behavior to the right for the graph of this function. Will the function be useful in modeling the number of Americans living with HIV and AIDS over an extended period of time? Explain your answer.

74. a. Use both functions to find the number of Americans living with HIV and AIDS in 2005. Which function provides a better description for the actual number shown in the bar graph?

b. Consider the function from part (a) that serves as a better model for 2005. Use the Leading Coefficient Test to determine the end behavior to the right for the graph of this function. Based on this end behavior, can the function be used to model the number of Americans living with HIV and AIDS over an extended period of time? Explain your answer.

75. During a diagnostic evaluation, a 33-year-old woman experienced a panic attack a few minutes after she had been asked to relax her whole body. The graph shows the rapid increase in heart rate during the panic attack.

Heart Rate before and during a Panic Attack



Source: Davis and Palladino, *Psychology*, Fifth Edition, Prentice Hall, 2007

a. For which time periods during the diagnostic evaluation was the woman's heart rate increasing?

b. For which time periods during the diagnostic evaluation was the woman's heart rate decreasing?

c. How many turning points (from increasing to decreasing or from decreasing to increasing) occurred for the woman's heart rate during the first 12 minutes of the diagnostic evaluation?

d. Suppose that a polynomial function is used to model the data displayed by the graph using

(time during the evaluation, heart rate).

Use the number of turning points to determine the degree of the polynomial function of best fit.

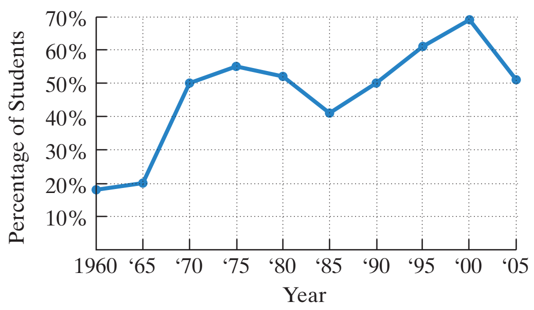
e. For the model in part (d), should the leading coefficient of the polynomial function be positive or negative? Explain your answer.

f. Use the graph to estimate the woman's maximum heart rate during the first 12 minutes of the diagnostic evaluation. After how many minutes did this occur?

g. Use the graph to estimate the woman's minimum heart rate during the first 12 minutes of the diagnostic evaluation. After how many minutes did this occur?

76. Even after a campaign to curb grade inflation, 51% of the grades given at Harvard in the 2005 school year were B+ or better. The graph shows the percentage of Harvard students with B+ averages or better for the period from 1960 through 2005.

Percentage of Harvard Students with B+ Averages or Better



Source: Mother Jones, January/February 2008

a. For which years was the percentage of students with B+ averages or better increasing?

b. For which years was the percentage of students with B+ averages or better decreasing?

c. How many turning points (from increasing to decreasing or from decreasing to increasing) does the graph have for the period shown?

d. Suppose that a polynomial function is used to model the data shown in the graph using

(number of years after 1960, percentage of students with B+ averages or better).

Use the number of turning points to determine the degree of the polynomial function of best fit.

e. For the model in part (d), should the leading coefficient of the polynomial function be positive or negative? Explain your answer.

f. Use the graph to estimate the maximum percentage of Harvard students with B+ averages or better. In which year did this occur?

g. Use the graph to estimate the minimum percentage of Harvard students with B+ averages or better. In which year did this occur?

Writing in Mathematics

77. What is a polynomial function?

78. What do we mean when we describe the graph of a polynomial function as smooth and continuous?

79. What is meant by the end behavior of a polynomial function?

80. Explain how to use the Leading Coefficient Test to determine the end behavior of a polynomial function.

81. Why is a third-degree polynomial function with a negative leading coefficient not appropriate for modeling nonnegative real-world phenomena over a long period of time?

82. What are the zeros of a polynomial function and how are they found?

83. Explain the relationship between the multiplicity of a zero and whether or not the graph crosses or touches the x-axis at that zero.

84. If f is a polynomial function, and $f(a)$ and $f(b)$ have opposite signs, what must occur between a and b? If $f(a)$ and $f(b)$ have the same sign, does it necessarily mean that this will not occur? Explain your answer.

85. Explain the relationship between the degree of a polynomial function and the number of turning points on its graph.

86. Can the graph of a polynomial function have no x-intercepts? Explain.

87. Can the graph of a polynomial function have no y-intercept? Explain.

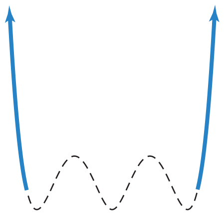
88. Describe a strategy for graphing a polynomial function. In your description, mention intercepts, the polynomial's degree, and turning points.

Technology Exercises

89. Use a graphing utility to verify any five of the graphs that you drew by hand in Exercises 41–64.

Write a polynomial function that imitates the end behavior of each graph in Exercises 90–93. The dashed portions of the graphs indicate that you should focus only on imitating the left and right behavior of the graph and can be flexible about what occurs between the left and right ends. Then use your graphing utility to graph the polynomial function and verify that you imitated the end behavior shown in the given graph.

90.



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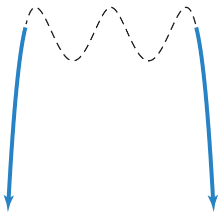

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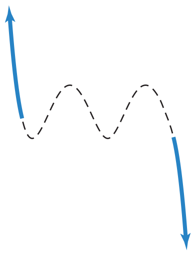

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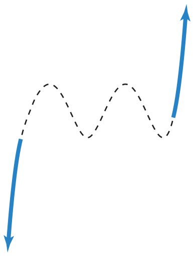

In Exercises 94–97, use a graphing utility with a viewing rectangle large enough to show end behavior to graph each polynomial function.

94. $f(x) = x^3 + 13x^2 + 10x - 4$

95. $f(x) = -2x^3 + 6x^2 + 3x - 1$

96. $f(x) = -x^4 + 8x^3 + 4x^2 + 2$

97. $f(x) = -x^5 + 5x^4 - 6x^3 + 2x + 20$

In Exercises 98–99, use a graphing utility to graph f and g in the same viewing rectangle. Then use the ZOOM OUT *feature to show that f and g have identical end behavior.*

98. $f(x) = x^3 - 6x + 1, \quad g(x) = x^3$

99. $f(x) = -x^4 + 2x^3 - 6x, \quad g(x) = -x^4$

Critical Thinking Exercises

Make Sense? *In Exercises 100–103, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

100. When I'm trying to determine end behavior, it's the coefficient of the leading term of a polynomial function that I should inspect.

101. I graphed $f(x) = (x + 2)^3(x - 4)^2$, and the graph touched the x-axis and turned around at -2.

102. I'm graphing a fourth-degree polynomial function with four turning points.

103. Although I have not yet learned techniques for finding the x-intercepts of $f(x) = x^3 + 2x^2 - 5x - 6$, I can easily determine the y-intercept.

In Exercises 104–107, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

104. If $f(x) = -x^3 + 4x$, then the graph of f falls to the left and falls to the right.

105. A mathematical model that is a polynomial of degree n whose leading term is $a_n x^n$, n odd and $a_n < 0$, is ideally suited to describe phenomena that have positive values over unlimited periods of time.

106. There is more than one third-degree polynomial function with the same three x-intercepts.

107. The graph of a function with origin symmetry can rise to the left and rise to the right.

Use the descriptions in Exercises 108–109 to write an equation of a polynomial function with the given characteristics. Use a graphing utility to graph your function to see if you are correct. If not, modify the function's equation and repeat this process.

108. Crosses the x-axis at -4, 0, and 3; lies above the x-axis between -4 and 0; lies below the x-axis between 0 and 3

109. Touches the x-axis at 0 and crosses the x-axis at 2; lies below the x-axis between 0 and 2

Preview Exercises

Exercises 110–112 will help you prepare for the material covered in the next section.

110. Divide 737 by 21 without using a calculator. Write the answer as

$$\text{quotient} + \frac{\text{remainder}}{\text{divisor}}.$$

111. Rewrite $4 - 5x - x^2 + 6x^3$ in descending powers of x.

112. Use

$$\frac{2x^3 - 3x^2 - 11x + 6}{x - 3} = 2x^2 + 3x - 2$$

to factor $2x^3 - 3x^2 - 11x + 6$ completely.