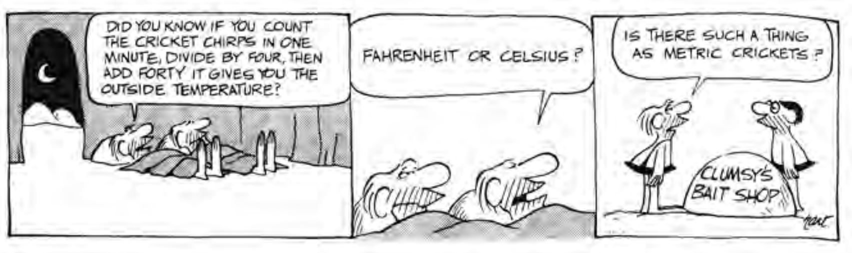
Exercises 89–90 are based on the following cartoon.



89. Assuming that there is no such thing as metric crickets, I modeled the information in the first frame of the cartoon with the function

$$T(n) = \frac{n}{4} + 40,$$

where $T(n)$ is the temperature, in degrees Fahrenheit, and n is the number of cricket chirps per minute.

90. I used the function in Exercise 89 and found an equation for $T^{-1}(n)$, which expresses the number of cricket chirps per minute as a function of Fahrenheit temperature.

In Exercises 91–94, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

91. The inverse of $\{(1, 4), (2, 7)\}$ is $\{(2, 7), (1, 4)\}$.

92. The function $f(x) = 5$ is one-to-one.

93. If $f(x) = 3x$, then $f^{-1}(x) = \frac{1}{3x}$.

94. The domain of f is the same as the range of f^{-1}.

95. If $f(x) = 3x$ and $g(x) = x + 5$, find $(f \circ g)^{-1}(x)$ and $(g^{-1} \circ f^{-1})(x)$.

96. Show that

$$f(x) = \frac{3x - 2}{5x - 3}$$

is its own inverse.

97. *Freedom 7* was the spacecraft that carried the first American into space in 1961. Total flight time was 15 minutes and the spacecraft reached a maximum height of 116 miles. Consider a function, s, that expresses *Freedom 7*'s height, $s(t)$, in miles, after t minutes. Is s a one-to-one function? Explain your answer.

98. If $f(2) = 6$, and f is one-to-one, find x satisfying $8 + f^{-1}(x - 1) = 10$.

Group Exercise

99. In Tom Stoppard's play *Arcadia*, the characters dream and talk about mathematics, including ideas involving graphing, composite functions, symmetry, and lack of symmetry in things that are tangled, mysterious, and unpredictable. Group members should read the play. Present a report on the ideas discussed by the characters that are related to concepts that we studied in this chapter. Bring in a copy of the play and read appropriate excerpts.

Preview Exercises

Exercises 100–102 will help you prepare for the material covered in the next section.

100. Let $(x_1, y_1) = (7, 2)$ and $(x_2, y_2) = (1, -1)$. Find $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. Express the answer in simplified radical form.

101. Use a rectangular coordinate system to graph the circle with center $(1, -1)$ and radius 1.

102. Solve by completing the square: $y^2 - 6y - 4 = 0$.

Section 1.9 Distance and Midpoint Formulas; Circles

Objectives

1. Find the distance between two points.
2. Find the midpoint of a line segment.
3. Write the standard form of a circle's equation.
4. Give the center and radius of a circle whose equation is in standard form.
5. Convert the general form of a circle's equation to standard form.

It's a good idea to know your way around a circle. Clocks, angles, maps, and compasses are based on circles. Circles occur everywhere in nature: in ripples on water, patterns on a moth's wings, and cross sections of trees. Some consider the circle to be the most pleasing of all shapes.

The rectangular coordinate system gives us a unique way of knowing a circle. It enables us to translate a circle's geometric definition into an algebraic equation. To do this, we must first develop a formula for the distance between any two points in rectangular coordinates.

1 Find the distance between two points.

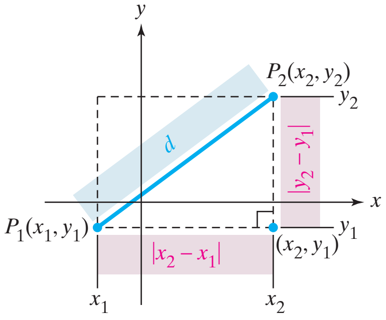


Figure 1.73

The Distance Formula

Using the Pythagorean Theorem, we can find the distance between the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in the rectangular coordinate system. The two points are illustrated in **Figure 1.73**.

The distance that we need to find is represented by d and shown in blue. Notice that the distance between the two points on the dashed horizontal line is the absolute value of the difference between the x-coordinates of the points. This distance, $|x_2 - x_1|$, is shown in pink. Similarly, the distance between the two points on the dashed vertical line is the absolute value of the difference between the y-coordinates of the points. This distance, $|y_2 - y_1|$, is also shown in pink.

Because the dashed lines in **Figure 1.73** are horizontal and vertical, a right triangle is formed. Thus, we can use the Pythagorean Theorem to find the distance d. Squaring the lengths of the triangle's sides results in positive numbers, so absolute value notation is not necessary.

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$ Apply the Pythagorean Theorem to the right triangle in Figure 1.72.

$$d = \pm\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$ Apply the square root property.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$ Because distance is nonnegative, write only the principal square root.

This result is called the **distance formula**.

When using the distance formula, it does not matter which point you call (x_1, y_1) and which you call (x_2, y_2).

The Distance Formula

The distance, d, between the points (x_1, y_1) and (x_2, y_2) in the rectangular coordinate system is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

To compute the distance between two points, find the square of the difference between the x-coordinates plus the square of the difference between the y-coordinates. The principal square root of this sum is the distance.

EXAMPLE 1 Using the Distance Formula

Find the distance between $(-1, 4)$ and $(3, -2)$.

Solution We will let $(x_1, y_1) = (-1, 4)$ and $(x_2, y_2) = (3, -2)$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$ Use the distance formula.

$$= \sqrt{[3 - (-1)]^2 + (-2 - 4)^2}$$ Substitute the given values.

$$= \sqrt{4^2 + (-6)^2}$$ Perform operations inside grouping symbols: $3 - (-1) = 3 + 1 = 4$ and $-2 - 4 = -6$.

$$= \sqrt{16 + 36}$$ Caution: This does not equal $\sqrt{16} + \sqrt{36}$. Square 4 and -6.

$$= \sqrt{52}$$ Add.

$$= \sqrt{4 \cdot 13} = 2\sqrt{13} \approx 7.21$$ $\sqrt{52} = \sqrt{4 \cdot 13} = \sqrt{4}\sqrt{13} = 2\sqrt{13}$

The distance between the given points is $2\sqrt{13}$ units, or approximately 7.21 units. The situation is illustrated in **Figure 1.74**.

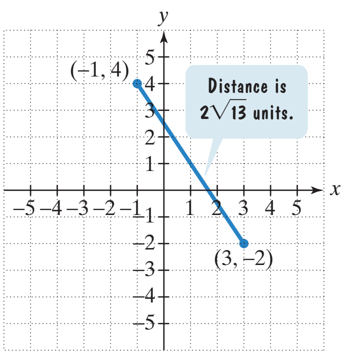


Figure 1.74 Finding the distance between two points

Check Point 1 Find the distance between $(-4, 9)$ and $(1, -3)$.

2 Find the midpoint of a line segment.

The Midpoint Formula

The distance formula can be used to derive a formula for finding the midpoint of a line segment between two given points. The formula is given as follows:

The Midpoint Formula

Consider a line segment whose endpoints are (x_1, y_1) and (x_2, y_2). The coordinates of the segment's midpoint are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

To find the midpoint, take the average of the two x-coordinates and the average of the two y-coordinates.

Study Tip

The midpoint formula requires finding the *sum* of coordinates. By contrast, the distance formula requires finding the *difference* of coordinates:

Midpoint: Sum of coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Distance: Difference of coordinates

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

It's easy to confuse the two formulas. Be sure to use addition, not subtraction, when applying the midpoint formula.

EXAMPLE 2 Using the Midpoint Formula

Find the midpoint of the line segment with endpoints $(1, -6)$ and $(-8, -4)$.

Solution To find the coordinates of the midpoint, we average the coordinates of the endpoints.

$$\text{Midpoint} = \left(\frac{1 + (-8)}{2}, \frac{-6 + (-4)}{2}\right) = \left(\frac{-7}{2}, \frac{-10}{2}\right) = \left(-\frac{7}{2}, -5\right)$$

Average the x-coordinates. Average the y-coordinates.

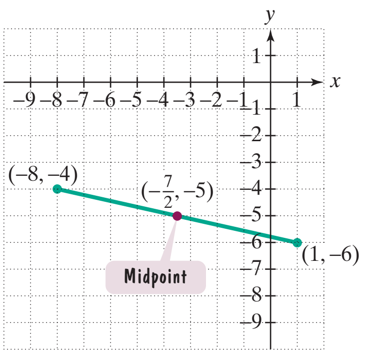


Figure 1.75 Finding a line segment's midpoint

Figure 1.75 illustrates that the point $\left(-\frac{7}{2}, -5\right)$ is midway between the points $(1, -6)$ and $(-8, -4)$.

Check Point 2 Find the midpoint of the line segment with endpoints $(1, 2)$ and $(7, -3)$.

Circles

Our goal is to translate a circle's geometric definition into an equation. We begin with this geometric definition.

Definition of a Circle

A **circle** is the set of all points in a plane that are equidistant from a fixed point, called the **center**. The fixed distance from the circle's center to any point on the circle is called the **radius**.

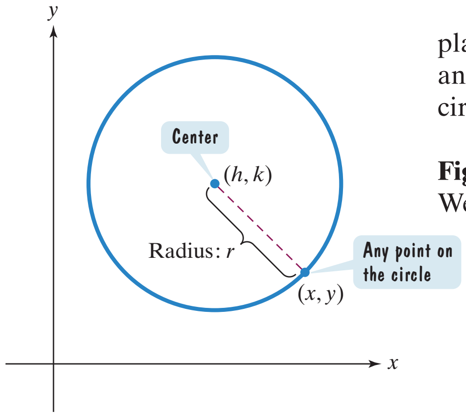


Figure 1.76 A circle centered at (h, k) with radius r

Figure 1.76 is our starting point for obtaining a circle's equation. We've placed the circle into a rectangular coordinate system. The circle's center is (h, k) and its radius is r. We let (x, y) represent the coordinates of any point on the circle.

What does the geometric definition of a circle tell us about the point (x, y) in **Figure 1.76**? The point is on the circle if and only if its distance from the center is r. We can use the distance formula to express this idea algebraically:

The distance between (x, y) and (h, k) — is always — r.

$$\sqrt{(x-h)^2+(y-k)^2} = r.$$

Squaring both sides of $\sqrt{(x-h)^2+(y-k)^2} = r$ yields the *standard form of the equation of a circle.*

3 Write the standard form of a circle's equation.

The Standard Form of the Equation of a Circle

The **standard form of the equation of a circle** with center (h, k) and radius r is

$$(x-h)^2+(y-k)^2=r^2.$$

EXAMPLE 3 Finding the Standard Form of a Circle's Equation

Write the standard form of the equation of the circle with center (0, 0) and radius 2. Graph the circle.

Solution The center is (0, 0). Because the center is represented as (h, k) in the standard form of the equation, $h = 0$ and $k = 0$. The radius is 2, so we will let $r = 2$ in the equation.

$$(x-h)^2+(y-k)^2=r^2 \quad \text{This is the standard form of a circle's equation.}$$

$$(x-0)^2+(y-0)^2=2^2 \quad \text{Substitute 0 for } h\text{, 0 for } k\text{, and 2 for } r.$$

$$x^2+y^2=4 \quad \text{Simplify.}$$

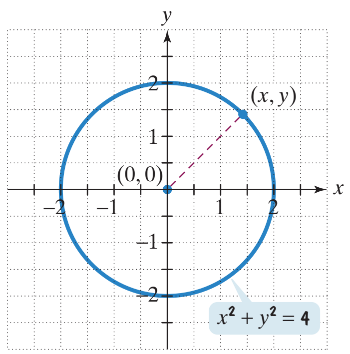
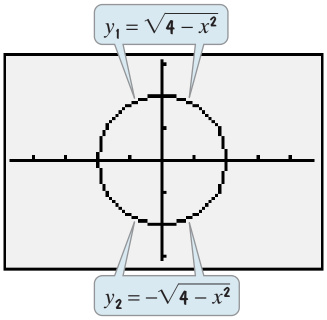


Figure 1.77 The graph of $x^2 + y^2 = 4$

The standard form of the equation of the circle is $x^2 + y^2 = 4$. **Figure 1.77** shows the graph.

Check Point 3 Write the standard form of the equation of the circle with center (0, 0) and radius 4.

Technology



To graph a circle with a graphing utility, first solve the equation for y.

$$x^2 + y^2 = 4$$
$$y^2 = 4 - x^2$$
$$y = \pm\sqrt{4 - x^2} \quad \text{y is not a function of x.}$$

Graph the two equations

$$y_1 = \sqrt{4 - x^2} \quad \text{and} \quad y_2 = -\sqrt{4 - x^2}$$

in the same viewing rectangle. The graph of $y_1 = \sqrt{4 - x^2}$ is the top semicircle because y is always positive. The graph of $y_2 = -\sqrt{4 - x^2}$ is the bottom semicircle because y is always negative. Use a ZOOM SQUARE setting so that the circle looks like a circle. (Many graphing utilities have problems connecting the two semicircles because the segments directly to the left and to the right of the center become nearly vertical.)

Example 3 and Check Point 3 involved circles centered at the origin. The standard form of the equation of all such circles is $x^2 + y^2 = r^2$, where r is the circle's radius. Now, let's consider a circle whose center is not at the origin.

EXAMPLE 4 Finding the Standard Form of a Circle's Equation

Write the standard form of the equation of the circle with center $(-2, 3)$ and radius 4.

Solution The center is $(-2, 3)$. Because the center is represented as (h, k) in the standard form of the equation, $h = -2$ and $k = 3$. The radius is 4, so we will let $r = 4$ in the equation.

$$(x - h)^2 + (y - k)^2 = r^2 \quad \text{This is the standard form of a circle's equation.}$$
$$[x - (-2)]^2 + (y - 3)^2 = 4^2 \quad \text{Substitute -2 for h, 3 for k, and 4 for r.}$$
$$(x + 2)^2 + (y - 3)^2 = 16 \quad \text{Simplify.}$$

The standard form of the equation of the circle is $(x + 2)^2 + (y - 3)^2 = 16$.

Check Point 4 Write the standard form of the equation of the circle with center $(0, -6)$ and radius 10.

4 Give the center and radius of a circle whose equation is in standard form.

EXAMPLE 5 Using the Standard Form of a Circle's Equation to Graph the Circle

a. Find the center and radius of the circle whose equation is

$$(x - 2)^2 + (y + 4)^2 = 9.$$

b. Graph the equation.

c. Use the graph to identify the relation's domain and range.

Solution

a. We begin by finding the circle's center, (h, k), and its radius, r. We can find the values for h, k, and r by comparing the given equation to the standard form of the equation of a circle, $(x - h)^2 + (y - k)^2 = r^2$.

$$(x - 2)^2 + (y + 4)^2 = 9$$
$$(x - 2)^2 + (y - (-4))^2 = 3^2$$

This is $(x - h)^2$, with $h = 2$. This is $(y - k)^2$, with $k = -4$. This is r^2, with $r = 3$.

We see that $h = 2$, $k = -4$, and $r = 3$. Thus, the circle has center $(h, k) = (2, -4)$ and a radius of 3 units.

b. To graph this circle, first locate the center $(2, -4)$. Because the radius is 3, you can locate at least four points on the circle by going out three units to the right, to the left, up, and down from the center.

The points three units to the right and to the left of $(2, -4)$ are $(5, -4)$ and $(-1, -4)$, respectively. The points three units up and down from $(2, -4)$ are $(2, -1)$ and $(2, -7)$, respectively.

Using these points, we obtain the graph in **Figure 1.78**.

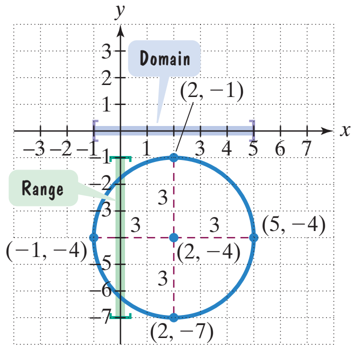


Figure 1.78 The graph of $(x - 2)^2 + (y + 4)^2 = 9$

c. The four points that we located on the circle can be used to determine the relation's domain and range. The points $(-1, -4)$ and $(5, -4)$ show that values of x extend from -1 to 5, inclusive:

$$\text{Domain} = [-1, 5].$$

The points $(2, -7)$ and $(2, -1)$ show that values of y extend from -7 to -1, inclusive:

$$\text{Range} = [-7, -1].$$

Study Tip

It's easy to make sign errors when finding h and k, the coordinates of a circle's center, (h, k). Keep in mind that h and k are the numbers that *follow the subtraction signs* in a circle's equation:

$$(x - 2)^2 + (y + 4)^2 = 9$$
$$(x - 2)^2 + (y - (-4))^2 = 9.$$

The number *after* the subtraction is 2: $h = 2$.

The number *after* the subtraction is -4: $k = -4$.

Check Point 5

a. Find the center and radius of the circle whose equation is

$$(x + 3)^2 + (y - 1)^2 = 4.$$

b. Graph the equation.

c. Use the graph to identify the relation's domain and range.

If we square $x - 2$ and $y + 4$ in the standard form of the equation in Example 5, we obtain another form for the circle's equation.

$$(x - 2)^2 + (y + 4)^2 = 9 \quad \text{This is the standard form of the equation in Example 5.}$$
$$x^2 - 4x + 4 + y^2 + 8y + 16 = 9 \quad \text{Square } x - 2 \text{ and } y + 4.$$
$$x^2 + y^2 - 4x + 8y + 20 = 9 \quad \text{Combine constants and rearrange terms.}$$
$$x^2 + y^2 - 4x + 8y + 11 = 0 \quad \text{Subtract 9 from both sides.}$$

This result suggests that an equation in the form $x^2 + y^2 + Dx + Ey + F = 0$ can represent a circle. This is called the *general form of the equation of a circle*.

The General Form of the Equation of a Circle

The **general form of the equation of a circle** is

$$x^2 + y^2 + Dx + Ey + F = 0,$$

where D, E, and F are real numbers.

5 Convert the general form of a circle's equation to standard form.

We can convert the general form of the equation of a circle to the standard form $(x - h)^2 + (y - k)^2 = r^2$. We do so by completing the square on x and y. Let's see how this is done.

Study Tip

To review completing the square, see Section P.7, pages 92–93.

EXAMPLE 6 Converting the General Form of a Circle's Equation to Standard Form and Graphing the Circle

Write in standard form and graph: $x^2 + y^2 + 4x - 6y - 23 = 0$.

Solution Because we plan to complete the square on both x and y, let's rearrange the terms so that x-terms are arranged in descending order, y-terms are arranged in descending order, and the constant term appears on the right.

$$x^2 + y^2 + 4x - 6y - 23 = 0$$ This is the given equation.

$$(x^2 + 4x) + (y^2 - 6y) = 23$$ Rewrite in anticipation of completing the square.

$$(x^2 + 4x + 4) + (y^2 - 6y + 9) = 23 + 4 + 9$$ Complete the square on x: $\frac{1}{2}\cdot 4 = 2$ and $2^2 = 4$, so add 4 to both sides. Complete the square on y: $\frac{1}{2}(-6) = -3$ and $(-3)^2 = 9$, so add 9 to both sides.

Remember that numbers added on the left side must also be added on the right side.

$$(x + 2)^2 + (y - 3)^2 = 36$$ Factor on the left and add on the right.

This last equation is in standard form. We can identify the circle's center and radius by comparing this equation to the standard form of the equation of a circle, $(x - h)^2 + (y - k)^2 = r^2$.

$$(x + 2)^2 + (y - 3)^2 = 36$$

$$(x - (-2))^2 + (y - 3)^2 = 6^2$$

This is $(x - h)^2$, with $h = -2$. This is $(y - k)^2$, with $k = 3$. This is r^2, with $r = 6$.

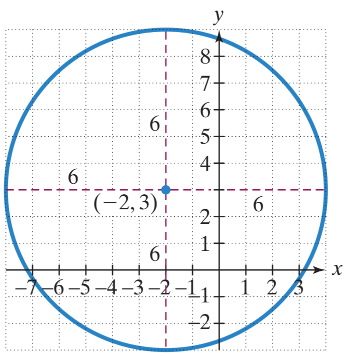


Figure 1.79 The graph of $(x + 2)^2 + (y - 3)^2 = 36$

We use the center, $(h, k) = (-2, 3)$, and the radius, $r = 6$, to graph the circle. The graph is shown in **Figure 1.79**.

Technology

To graph $x^2 + y^2 + 4x - 6y - 23 = 0$, the general form of the circle's equation given in Example 6, rewrite the equation as a quadratic equation in y.

$$y^2 - 6y + (x^2 + 4x - 23) = 0$$

Now solve for y using the quadratic formula, with $a = 1$, $b = -6$, and $c = x^2 + 4x - 23$.

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-6) \pm \sqrt{(-6)^2 - 4\cdot 1(x^2 + 4x - 23)}}{2\cdot 1} = \frac{6 \pm \sqrt{36 - 4(x^2 + 4x - 23)}}{2}$$

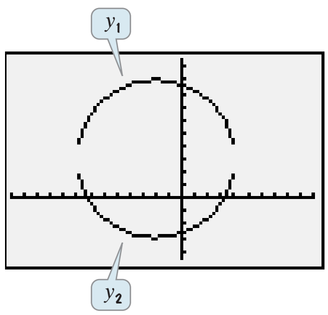
Because we will enter these equations, there is no need to simplify. Enter

$$y_1 = \frac{6 + \sqrt{36 - 4(x^2 + 4x - 23)}}{2}$$

and

$$y_2 = \frac{6 - \sqrt{36 - 4(x^2 + 4x - 23)}}{2}.$$

Use a [ZOOM SQUARE] setting. The graph is shown on the right.



Check Point 6 Write in standard form and graph:

$$x^2 + y^2 + 4x - 4y - 1 = 0.$$

Exercise Set 1.9

Practice Exercises

In Exercises 1–18, find the distance between each pair of points. If necessary, round answers to two decimals places.

1. $(2, 3)$ and $(14, 8)$
2. $(5, 1)$ and $(8, 5)$
3. $(4, -1)$ and $(-6, 3)$
4. $(2, -3)$ and $(-1, 5)$
5. $(0, 0)$ and $(-3, 4)$
6. $(0, 0)$ and $(3, -4)$
7. $(-2, -6)$ and $(3, -4)$
8. $(-4, -1)$ and $(2, -3)$
9. $(0, -3)$ and $(4, 1)$
10. $(0, -2)$ and $(4, 3)$
11. $(3.5, 8.2)$ and $(-0.5, 6.2)$
12. $(2.6, 1.3)$ and $(1.6, -5.7)$
13. $(0, -\sqrt{3})$ and $(\sqrt{5}, 0)$
14. $(0, -\sqrt{2})$ and $(\sqrt{7}, 0)$
15. $(3\sqrt{3}, \sqrt{5})$ and $(-\sqrt{3}, 4\sqrt{5})$
16. $(2\sqrt{3}, \sqrt{6})$ and $(-\sqrt{3}, 5\sqrt{6})$
17. $\left(\frac{7}{3}, \frac{1}{5}\right)$ and $\left(\frac{1}{3}, \frac{6}{5}\right)$
18. $\left(-\frac{1}{4}, -\frac{1}{7}\right)$ and $\left(\frac{3}{4}, \frac{6}{7}\right)$

In Exercises 19–30, find the midpoint of each line segment with the given endpoints.

19. $(6, 8)$ and $(2, 4)$
20. $(10, 4)$ and $(2, 6)$
21. $(-2, -8)$ and $(-6, -2)$
22. $(-4, -7)$ and $(-1, -3)$
23. $(-3, -4)$ and $(6, -8)$
24. $(-2, -1)$ and $(-8, 6)$
25. $\left(-\frac{7}{2}, \frac{3}{2}\right)$ and $\left(-\frac{5}{2}, -\frac{11}{2}\right)$
26. $\left(-\frac{2}{5}, \frac{7}{15}\right)$ and $\left(-\frac{2}{5}, -\frac{4}{15}\right)$
27. $(8, 3\sqrt{5})$ and $(-6, 7\sqrt{5})$
28. $(7\sqrt{3}, -6)$ and $(3\sqrt{3}, -2)$
29. $(\sqrt{18}, -4)$ and $(\sqrt{2}, 4)$
30. $(\sqrt{50}, -6)$ and $(\sqrt{2}, 6)$

In Exercises 31–40, write the standard form of the equation of the circle with the given center and radius.

31. Center $(0, 0), r = 7$
32. Center $(0, 0), r = 8$
33. Center $(3, 2), r = 5$
34. Center $(2, -1), r = 4$
35. Center $(-1, 4), r = 2$
36. Center $(-3, 5), r = 3$
37. Center $(-3, -1), r = \sqrt{3}$
38. Center $(-5, -3), r = \sqrt{5}$
39. Center $(-4, 0), r = 10$
40. Center $(-2, 0), r = 6$

In Exercises 41–52, give the center and radius of the circle described by the equation and graph each equation. Use the graph to identify the relation's domain and range.

41. $x^2 + y^2 = 16$
42. $x^2 + y^2 = 49$
43. $(x - 3)^2 + (y - 1)^2 = 36$
44. $(x - 2)^2 + (y - 3)^2 = 16$
45. $(x + 3)^2 + (y - 2)^2 = 4$
46. $(x + 1)^2 + (y - 4)^2 = 25$
47. $(x + 2)^2 + (y + 2)^2 = 4$
48. $(x + 4)^2 + (y + 5)^2 = 36$
49. $x^2 + (y - 1)^2 = 1$
50. $x^2 + (y - 2)^2 = 4$
51. $(x + 1)^2 + y^2 = 25$
52. $(x + 2)^2 + y^2 = 16$

In Exercises 53–64, complete the square and write the equation in standard form. Then give the center and radius of each circle and graph the equation.

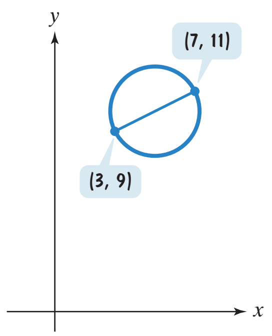
53. $x^2 + y^2 + 6x + 2y + 6 = 0$
54. $x^2 + y^2 + 8x + 4y + 16 = 0$
55. $x^2 + y^2 - 10x - 6y - 30 = 0$
56. $x^2 + y^2 - 4x - 12y - 9 = 0$
57. $x^2 + y^2 + 8x - 2y - 8 = 0$
58. $x^2 + y^2 + 12x - 6y - 4 = 0$
59. $x^2 - 2x + y^2 - 15 = 0$
60. $x^2 + y^2 - 6y - 7 = 0$
61. $x^2 + y^2 - x + 2y + 1 = 0$
62. $x^2 + y^2 + x + y - \frac{1}{2} = 0$
63. $x^2 + y^2 + 3x - 2y - 1 = 0$
64. $x^2 + y^2 + 3x + 5y + \frac{9}{4} = 0$

Practice Plus

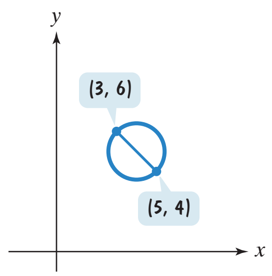
In Exercises 65–66, a line segment through the center of each circle intersects the circle at the points shown.

a. *Find the coordinates of the circle's center.*
b. *Find the radius of the circle.*
c. *Use your answers from parts (a) and (b) to write the standard form of the circle's equation.*

65.



66.



In Exercises 67–70, graph both equations in the same rectangular coordinate system and find all points of intersection. Then show that these ordered pairs satisfy the equations.

67. $x^2 + y^2 = 16$
$x - y = 4$

68. $x^2 + y^2 = 9$
$x - y = 3$

69. $(x - 2)^2 + (y + 3)^2 = 4$
$y = x - 3$

70. $(x - 3)^2 + (y + 1)^2 = 9$
$y = x - 1$

Application Exercises

The cell phone screen shows coordinates of six cities from a rectangular coordinate system placed on North America by long-distance telephone companies. Each unit in this system represents $\sqrt{0.1}$ mile.

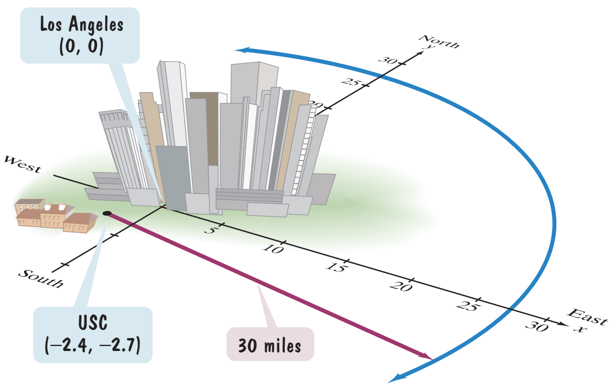
Source: Peter H. Dana

In Exercises 71–72, use this information to find the distance, to the nearest mile, between each pair of cities.

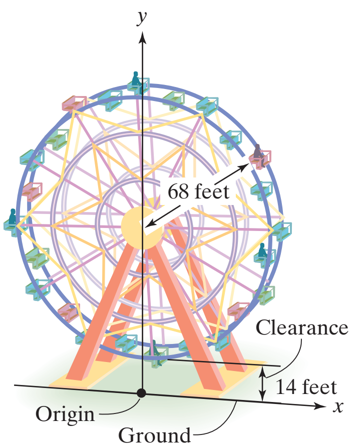
71. Boston and San Francisco

72. New Orleans and Houston

73. A rectangular coordinate system with coordinates in miles is placed with the origin at the center of Los Angeles. The figure indicates that the University of Southern California is located 2.4 miles west and 2.7 miles south of central Los Angeles. A seismograph on the campus shows that a small earthquake occurred. The quake's epicenter is estimated to be approximately 30 miles from the university. Write the standard form of the equation for the set of points that could be the epicenter of the quake.



74. The Ferris wheel in the figure has a radius of 68 feet. The clearance between the wheel and the ground is 14 feet. The rectangular coordinate system shown has its origin on the ground directly below the center of the wheel. Use the coordinate system to write the equation of the circular wheel.



Writing in Mathematics

75. In your own words, describe how to find the distance between two points in the rectangular coordinate system.

76. In your own words, describe how to find the midpoint of a line segment if its endpoints are known.

77. What is a circle? Without using variables, describe how the definition of a circle can be used to obtain a form of its equation.

78. Give an example of a circle's equation in standard form. Describe how to find the center and radius for this circle.

79. How is the standard form of a circle's equation obtained from its general form?

80. Does $(x - 3)^2 + (y - 5)^2 = 0$ represent the equation of a circle? If not, describe the graph of this equation.

81. Does $(x - 3)^2 + (y - 5)^2 = -25$ represent the equation of a circle? What sort of set is the graph of this equation?

82. Write and solve a problem about the flying time between a pair of cities shown on the cell phone screen for Exercises 71–72. Do not use the pairs in Exercise 71 or Exercise 72. Begin by determining a reasonable average speed, in miles per hour, for a jet flying between the cities.

Technology Exercises

In Exercises 83–85, use a graphing utility to graph each circle whose equation is given.

83. $x^2 + y^2 = 25$

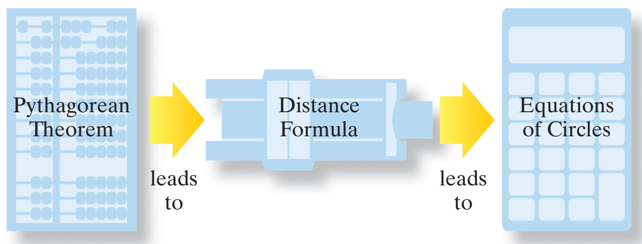
84. $(y + 1)^2 = 36 - (x - 3)^2$

85. $x^2 + 10x + y^2 - 4y - 20 = 0$

Critical Thinking Exercises

Make Sense? *In Exercises 86–89, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

86. I've noticed that in mathematics, one topic often leads logically to a new topic:



87. To avoid sign errors when finding h and k, I place parentheses around the numbers that follow the subtraction signs in a circle's equation.

88. I used the equation $(x + 1)^2 + (y - 5)^2 = -4$ to identify the circle's center and radius.

89. My graph of $(x - 2)^2 + (y + 1)^2 = 16$ is my graph of $x^2 + y^2 = 16$ translated two units right and one unit down.

In Exercises 90–93, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

90. The equation of the circle whose center is at the origin with radius 16 is $x^2 + y^2 = 16$.

91. The graph of $(x - 3)^2 + (y + 5)^2 = 36$ is a circle with radius 6 centered at $(-3, 5)$.

92. The graph of $(x - 4) + (y + 6) = 25$ is a circle with radius 5 centered at $(4, -6)$.

93. The graph of $(x - 3)^2 + (y + 5)^2 = -36$ is a circle with radius 6 centered at $(3, -5)$.

94. Show that the points $A(1, 1 + d)$, $B(3, 3 + d)$, and $C(6, 6 + d)$ are collinear (lie along a straight line) by showing that the distance from A to B plus the distance from B to C equals the distance from A to C.

95. Prove the midpoint formula by using the following procedure.

a. Show that the distance between (x_1, y_1) and $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ is equal to the distance between (x_2, y_2) and $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

b. Use the procedure from Exercise 94 and the distances from part (a) to show that the points (x_1, y_1), $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$, and (x_2, y_2) are collinear.

96. Find the area of the donut-shaped region bounded by the graphs of $(x - 2)^2 + (y + 3)^2 = 25$ and $(x - 2)^2 + (y + 3)^2 = 36$.

97. A **tangent line** to a circle is a line that intersects the circle at exactly one point. The tangent line is perpendicular to the radius of the circle at this point of contact. Write an equation in point-slope form for the line tangent to the circle whose equation is $x^2 + y^2 = 25$ at the point $(3, -4)$.

Preview Exercises

Exercises 98–100 will help you prepare for the material covered in the next section.

98. Write an algebraic expression for the fare increase if a \$200 plane ticket is increased to x dollars.

99. Find the perimeter and the area of each rectangle with the given dimensions:

a. 40 yards by 30 yards **b.** 50 yards by 20 yards.

100. Solve for h: $\pi r^2 h = 22$. Then rewrite $2\pi r^2 + 2\pi rh$ in terms of r.

Section 1.10 Modeling with Functions

Objectives

1. Construct functions from verbal descriptions.
2. Construct functions from formulas.

A can of Coca-Cola is sold every six seconds throughout the world.

In 2005, to curb consumption of sugared soda, the Center for Science in the Public Interest (CSPI) urged the FDA to slap cigarette-style warning labels on these drinks, calling them "liquid candy." Despite the variety of its reputations throughout the world, the soft drink industry has spent far more time reducing the amount of aluminum in its cylindrical cans than addressing the problems of the nutritional disaster floating within its packaging. In the 1960s, one pound of aluminum made fewer than 20 cans; today, almost 30 cans come out of the same amount. The thickness of the can wall is less than five-thousandths of an inch, about the same as a magazine cover.

Many real-world problems involve constructing mathematical models that are functions. The problem of minimizing the amount of aluminum needed to manufacture a 12-ounce soft-drink can first requires that we express the surface area of all such cans as a function of their radius. In constructing such a function, we must be able to translate a verbal description into a mathematical representation—that is, a mathematical model.

Study Tip

In calculus, you will solve problems involving maximum or minimum values of functions. Such problems often require creating the function that is to be maximized or minimized. Quite often, the calculus is fairly routine. It is the algebraic setting up of the function that causes difficulty. That is why the material in this section is so important.