Example 4 Find the exact value of $\cos 15°$.

Solution Use identity (18) with $\frac{x}{2} = 15°$ and thus, with $x = 30°$.

$$\cos 15° = +\sqrt{\frac{1 + \cos 30°}{2}}$$

$$= \sqrt{\frac{1 + \frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2 + \sqrt{3}}{4}} = \frac{\sqrt{2 + \sqrt{3}}}{2} \quad \blacksquare$$

To see that the answer in Example 4 is equivalent to the answer we found in Example 1 of Section 3-3, note that $\frac{2 + \sqrt{3}}{4} = \left(\frac{\sqrt{6} + \sqrt{2}}{4}\right)^2$.

EXERCISES 3-4

For each angle θ satisfying the given condition in the given quadrant, find (a) $\sin 2\theta$ and (b) $\cos 2\theta$.

A 1. $\sin \theta = \frac{4}{5}$; I

2. $\cos \theta = \frac{12}{13}$; I

3. $\cos \theta = -\frac{5}{13}$; II

4. $\sin \theta = -\frac{3}{5}$; III

5. $\sin \theta = -\frac{3}{4}$; III

6. $\cos \theta = \frac{2}{3}$; IV

7. $\cos \theta = -\frac{1}{3}$; II

8. $\sin \theta = -\frac{1}{4}$; IV

9. $\cos \theta = \frac{1}{5}$; IV

Exercises 10–18: For the angles in Exercises 1–9, find $\cos 4\theta$.

In Exercises 19–24, use a half-angle formula to evaluate each expression.

19. $\sin 15°$

20. $\cos 75°$

21. $\cos 67.5°$

22. $\sin 157.5°$

23. $\sin 75°$

24. $\sin 67.5°$

25. $\cos \frac{x}{2}$, if $\cos x = -\frac{8}{25}$ and $0 < x < \pi$

26. $\sin \frac{x}{2}$, if $\cos x = -\frac{31}{49}$ and $\pi < x < 2\pi$

27. $\sin \frac{x}{2}$, if $\sin x = \frac{24}{25}$ and $0 < x < \frac{\pi}{2}$

28. $\cos \frac{x}{2}$, if $\sin x = -\frac{4\sqrt{5}}{9}$ and $\frac{3\pi}{2} < x < 2\pi$

29. Derive identity (19) from identity (17a).

Prove each of the following identities.

B 30. $\csc x - 2\sin x = \csc x \cos 2x$

31. $\dfrac{\sin 2x}{1 - \cos 2x} = \cot x$

32. $\dfrac{1 - \cos 2\theta}{\cos 2\theta + 1} = \tan^2 \theta$

33. $\dfrac{1 - \cos 2\theta}{\sin \theta \sin 2\theta} = \sec \theta$

34. $1 - \tan^4 x = \dfrac{\cos 2x}{\cos^4 x}$

35. $\tan x + \tan y = \dfrac{\sin (x + y)}{\cos x \cos y}$

36. $(\sin \theta + \cos \theta)^2 = 1 + \sin 2\theta$

37. $\cos^4 \theta - \sin^4 \theta = \cos 2\theta$

38. $\cot \theta - \tan \theta = 2 \cot 2\theta$ $\left(\text{Hint: } \cot 2\theta = \dfrac{\cos 2\theta}{\sin 2\theta}\right)$

39. $\cot \theta + \tan \theta = 2 \csc 2\theta$

40. $\sin 4x = 4 \sin x \cos^3 x - 4 \sin^3 x \cos x$

41. $\cos 4x = 8 \cos^4 x - 8 \cos^2 x + 1$

42. $\cot 2\theta = \dfrac{1}{2}\left(\cot \theta - \dfrac{1}{\cot \theta}\right)$

43. $\tan \dfrac{x}{2} = \pm\sqrt{\dfrac{1 - \cos x}{1 + \cos x}}$

C 44. $\dfrac{\cos 2x + \sin^4 x}{\sin 2x} = \dfrac{1}{2} \cot x \cos^2 x$

45. $\sec 2x = \dfrac{\sec^2 x}{2 - \sec^2 x}$

46. $\csc^2 2x = \dfrac{\csc^4 x}{4(\csc^2 x - 1)}$

47. $\dfrac{\cos x + \sin x}{\cos x - \sin x} = \dfrac{1 + \sin 2x}{\cos 2x}$

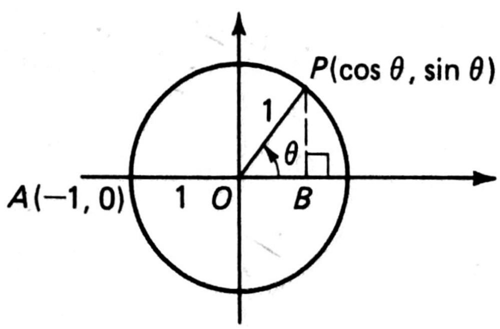
Exercise 48 outlines a way to derive the formula for $\cos \frac{\theta}{2}$ for an acute angle θ. It is thought to be similar to the method used by Hipparchus of Nicaea (page 1) to construct the first known table of trigonometric functions in intervals of 7.5°.

48. (a) Explain why $\angle PAB = \dfrac{\theta}{2}$.

(b) Explain why $\cos \angle PAB = \dfrac{1 + \cos \theta}{PA}$.

(c) Use the distance formula to find PA in simplest radical form.

(d) Substitute the value for PA found in (c) in the equation in (b) to get the formula $\cos \dfrac{\theta}{2} = \sqrt{\dfrac{1 + \cos \theta}{2}}$.



49. Explain how one could compute the values of $\cos \theta$ and $\sin \theta$ for θ in intervals of 7.5° between 0° and 90°, using only:

1. the half-angle formulas
2. the values of sine and cosine of 30° and 45°
3. the identities $\sin (90° - \theta) = \cos \theta$ and $\cos (90° - \theta) = \sin \theta$