1.6 Transformations of Functions

Objectives
1. Recognize graphs of common functions.
2. Use vertical shifts to graph functions.
3. Use horizontal shifts to graph functions.
4. Use reflections to graph functions.
5. Use vertical stretching and shrinking to graph functions.
6. Use horizontal stretching and shrinking to graph functions.
7. Graph functions involving a sequence of transformations.

Have you seen Terminator 2, The Mask, or The Matrix? These were among the first films to use spectacular effects in which a character or object having one shape was transformed in a fluid fashion into a quite different shape. The name for such a transformation is morphing. The effect allows a real actor to be seamlessly transformed into a computer-generated animation. The animation can be made to perform impossible feats before it is morphed back to the conventionally filmed image.

Like transformed movie images, the graph of one function can be turned into the graph of a different function. To do this, we need to rely on a function’s equation. Knowing that a graph is a transformation of a familiar graph makes graphing easier.

Table 1.3 Algebra’s Common Graphs

<table>
<thead>
<tr>
<th>Function Type</th>
<th>Equation</th>
<th>Domain</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant Function</td>
<td>( f(x) = c )</td>
<td>((-\infty, \infty))</td>
<td>c</td>
</tr>
<tr>
<td>Identity Function</td>
<td>( f(x) = x )</td>
<td>((-\infty, \infty))</td>
<td>((-\infty, \infty))</td>
</tr>
<tr>
<td>Absolute Value Function</td>
<td>( f(x) =</td>
<td>x</td>
<td>)</td>
</tr>
<tr>
<td>Standard Quadratic Function</td>
<td>( f(x) = x^2 )</td>
<td>((-\infty, \infty))</td>
<td>[0, \infty)</td>
</tr>
<tr>
<td>Square Root Function</td>
<td>( f(x) = \sqrt{x} )</td>
<td>([0, \infty))</td>
<td>[0, \infty)</td>
</tr>
<tr>
<td>Standard Cubic Function</td>
<td>( f(x) = x^3 )</td>
<td>((-\infty, \infty))</td>
<td>((-\infty, \infty))</td>
</tr>
<tr>
<td>Cube Root Function</td>
<td>( f(x) = \sqrt[3]{x} )</td>
<td>((-\infty, \infty))</td>
<td>((-\infty, \infty))</td>
</tr>
</tbody>
</table>
**EXAMPLE 1**  **Vertical Shift Downward**

Use the graph of \( f(x) = |x| \) to obtain the graph of \( g(x) = |x| - 4 \).

![Graph of \( f(x) = |x| \) and \( g(x) = |x| - 4 \)]

**Check Point**  Use the graph of \( f(x) = |x| \) to obtain the graph of \( g(x) = |x| + 3 \).

![Graph of \( f(x) = |x| \) and \( g(x) = |x| + 3 \)]

**Horizontal Shifts**

Let \( f \) be a function and \( c \) a positive real number.

- The graph of \( y = f(x + c) \) is the graph of \( y = f(x) \) shifted to the left \( c \) units.
- The graph of \( y = f(x - c) \) is the graph of \( y = f(x) \) shifted to the right \( c \) units.

![Graphs showing horizontal shifts](combined_horizontal_shifts.png)

**EXAMPLE 2**  **Horizontal Shift to the Left**

Use the graph of \( f(x) = \sqrt{x} \) to obtain the graph of \( g(x) = \sqrt{x + 5} \).

![Graphs showing horizontal shift](graph_of_root_function.png)

**EXAMPLE 3**  **Combining Horizontal and Vertical Shifts**

Use the graph of \( f(x) = x^2 \) to obtain the graph of \( h(x) = (x + 1)^2 - 3 \).
Reflection about the $x$-Axis

The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected about the $x$-axis.

**EXAMPLE 4** Reflection about the $x$-Axis

Use the graph of $f(x) = \sqrt{x}$ to obtain the graph of $g(x) = -\sqrt{x}$.

**Check Point 4** Use the graph of $f(x) = |x|$ to obtain the graph of $g(x) = -|x|$.

Reflection about the $y$-Axis

The graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected about the $y$-axis.

**EXAMPLE 5** Reflection about the $y$-Axis

Use the graph of $f(x) = \sqrt{x}$ to obtain the graph of $h(x) = \sqrt{-x}$.

**Check Point 5** Use the graph of $f(x) = \sqrt{x}$ to obtain the graph of $h(x) = \sqrt{-x}$.
**Example 6** Vertically Shrinking a Graph

Use the graph of \( f(x) = x^3 \) to obtain the graph of \( h(x) = \frac{1}{2}x^3 \).

**Check Point 6** Use the graph of \( f(x) = |x| \) to obtain the graph of \( g(x) = 2|x| \).

**Example 7** Horizontally Stretching and Shrinking a Graph

Use the graph of \( y = f(x) \) in Figure 1.59 to obtain each of the following graphs:

a. \( g(x) = f(2x) \)  
b. \( h(x) = f\left(\frac{1}{2}x\right) \).

*Figure 1.58* Vertically stretching and shrinking \( f(x) = x^2 \)

*Figure 1.59*
**Check Point 7** Use the graph of $y = f(x)$ in Figure 1.60 to obtain each of the following graphs:

a. $g(x) = f(2x)$  

b. $h(x) = f\left(\frac{x}{2}\right)$.

![Figure 1.60](image)

**Example 8** Graphing Using a Sequence of Transformations

Use the graph of $y = f(x)$ given in Figure 1.59 of Example 7 on page 212, and repeated below, to graph $y = -\frac{1}{2}f(x - 1) + 3$.

![Figure 1.59](image)
EXAMPLE 9 Graphing Using a Sequence of Transformations

Use the graph of $f(x) = x^2$ to graph $g(x) = 2(x + 3)^2 - 1$. 

1. **Horizontal shifting:** Graph by shifting the graph of $f(x) = x^2$ three units to the left.

2. **Stretching:** Graph by stretching the previous graph by a factor of 2.

3. **Vertical shifting:** Graph by shifting the previous graph down 1 unit.

$g(x) = 2(x + 3)^2 - 1$